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THE

PRINCIPLES

OF

MATHEMATICS

AND

NATURAL PHILOSOPHY.

IN FOUR VOLUMES.

VOL. III. PART I.

82529

THE PRINCIPLES OF MECHANICS:

DESIGNED

FOR THE USE OF STUDENTS

IN THE

UNIVERSITY.

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CAMBRIDGE,

PRINTED BY J. BURGES PRINTER TO THE UNIVERSITY;
AND SOLD BY J. DEIGHTON, J. NICHOLSON, AND W. H. LUNN,
CAMBRIDGE: F. WINGRAVE, H. GARDNER, AND
P. EMSLEY, IN THE STRAND; B. & J. WHITE,
FLEETSTREET; F. & C. RIVINGTON, AND
G. & T. WILKIE, IN ST. PAUL'S
CHURCHYARD, LONDON.

MDCCXCVI.

THE

PRINCIPLES OF MECHANICS.

THE term *Mechanics* has at different times, and by different writers, been applied to branches of ſcience eſſentially diſtinct from each other. It was originally confined to the doctrine of equilibrium, or the inveſtigation of the proportion of powers when they balance each other.

Later writers, adapting the term to their diſcoveries, have uſed it to denote that ſcience which treats of the nature, geneſis, and alteration of motion; giving to the former branch, by way of contradiſtinction, the name of ſtatics.

Others, giving the term a ſtill more comprehenſive meaning, have applied it to both theſe ſciences.

None of theſe definitions will exactly ſuit our preſent purpoſe; the firſt being too contracted, and the others much too extenſive, for a treatiſe which is intended to be an introduction to the higher branches of philoſophy. Our ſyſtem of mechanics will compriſe the doctrine of equilibrium, and ſo much of the ſcience of motion as is neceſſary to explain the effects of impact and gravity.

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SECTION

SECTION I.

ON MATTER AND MOTION.

DEFINITIONS.

ART. 1. *MATTER* is a ſubſtance, the object of our ſenſes, in which are always united the following properties; *extenſion*, *figure*, *ſolidity*, *mobility*, *diviſibility*, *gravity*, and *inactivity*.

2. *Extenſion* may be conſidered in three points of view: 1ſt. As ſimply denoting the part of ſpace which lies between two points, in which caſe it is called *diſtance*. 2d. As implying both length and breadth, when it is denominated *ſurface* or *area*. 3d. As compriſing three dimenſions, length, breadth, and thickneſs, in which caſe it may be called *capacity* or *content*. It is uſed in the laſt of theſe ſenſes when it is ſaid to be a property of matter.

3. *Figure* is the boundary of extenſion. The portions of matter, from which we receive our ideas of this ſubſtance, are bounded, or have figure.

4. *Solidity* is that property of matter by which it fills ſpace; or, by which any portion of matter ex-

cludes every other portion from that part of ſpace which it occupies; and thus it is capable of reſiſtance and protruſion. "There is no idea which we receive more conſtantly from ſenſation than *ſolidity*. Whether we move or reſt, in what poſture ſoever we are, we always feel ſomething under us that ſupports us, and hinders our farther ſinking downwards; and the bodies which we daily handle make us perceive that, whilſt they remain between them, they do by an inſurmountable force hinder the approach of the parts of our hands that preſs them." *

5. *Mobility*, or a capacity of being transferred from one place to another, is a quality found to belong to all bodies upon which we can make ſuitable experiments; and hence we conclude that it belongs to all matter.

6. *Diviſibility* ſignifies a capacity of being ſeparated into parts. That matter is thus diviſible, our daily experience aſſures us. How far the diviſion can actually be carried is not ſo eaſily ſeen. We know that many bodies may be reduced to a very fine powder by trituration; by chemical ſolution, the parts of a body may be ſo far divided as not to be ſenſible to the ſight; and by the help of the microſcope we diſcover myriads of organized bodies, totally unknown before ſuch inſtruments were invented. We are led, by ſuch conſiderations as theſe, to conclude, that the diviſion of matter is carried to a degree of minuteneſs far exceeding the bounds of our faculties; and it ſeems not unreaſonable to ſuppoſe, that this capacity of diviſion is without limit; eſpecially, as we can

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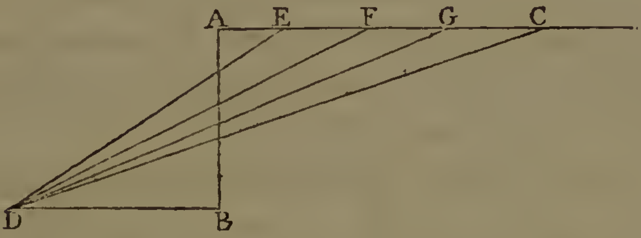
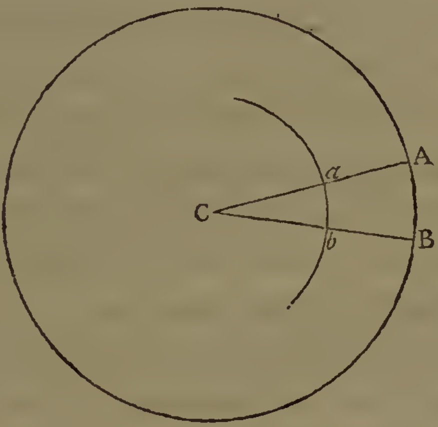
* Locke's Eſſay. B. II. Ch. IV.

prove that any portion of extenſion is diviſible into parts leſs and leſs without end.*

From the extremities of the line AB, draw AC, BD, parallel to each other, and in oppoſite directions; in AC take any number of points E, F, G, &c. and join DE, DF, DG, &c these lines will cut AB in different points; and ſince in the indefinite line AC an unlimited number of points may be taken, the number of parts into which AB is diviſible, is indefinite.



This property of extenſion may alſo be proved ex abſurdo. If poſſible, let AB be the leaſt portion of a circular



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* Porro corporum partes diviſas et ſibi mutuò contiguas ab invicem ſeparari poſſe ex phænomenis novimus, et partes indiviſas in partes minores ratione diſtingui poſſe ex mathematicâ certum eſt.

circular arc; take C the center, join AC, CB, and with the center C, and radius Ca, which is leſs than CA, deſcribe a circle cutting CA and CB in the points a and b; then becauſe AB and ab are ſimilar arcs, they are as their radii; therefore ab is leſs than AB; or a portion of extenſion leſs than the leaſt poſſible has been found, which is abſurd. Hence, any portion of extenſion is diviſible into parts leſs and leſs, without ever coming to a limit.

It has been ſuppoſed by ſome writers that there are certain indiviſible particles of matter, of the ſame form and dimenſions, by the different modifications of which different bodies are formed. This is a gratuitous aſſumption, unſupported by experiment; nor can it's truth or falſehood be brought to this teſt. And as the contrary is at leaſt poſſible, we cannot be certain, that concluſions founded on this hypotheſis, are juſt and practical.

7. *Gravity* is the tendency which all bodies have to the center of the earth.

We are convinced of the exiſtence of this tendency by obſerving, that, whenever a body is ſuſtained, it's preſſure is exerted in a direction perpendicular to the horizon; and that, when the impediment is removed, it always deſcends in that direction.

The *weight* of a body is it's tendency to the earth, compared with the like tendency of ſome other body, which is conſidered as a ſtandard. Thus, if a body with a certain degree of gravity be called one pound, any other body which has the ſame degree of gravity, or

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eſt. Utrum verò partes illæ diſtinctæ et nondum diviſæ per vires naturæ dividi et ab invicem ſeparari poſſint, incertum eſt. NEWT. Princip. L. III. Reg. 3.

or which by it's gravity will produce the ſame effect, under the ſame circumſtances, is alſo called a pound; and theſe two together, two pounds, &c.

Gravity is not an accidental property of matter ariſing from the figure or diſpoſition of the parts of a body; for then, by changing the ſhape, or altering the arrangement of the particles, the gravitation of the maſs would be altered. But we find that no ſeparation of the particles, no change of the ſtructure, which human power can effect, produces any alteration in the weight.

As gravity is a property belonging to every particle of a body, independent of it's ſituation with reſpect to other particles, the gravity of the whole is the aggregate of the gravities of all it's parts. Thus, though the weight of the whole is not altered by any diviſion, or new arrangement of the particles, yet every increaſe or diminution of their number, produces a correſponding increaſe or diminution of the weight.

Our preſent ſubject does not lead us to conſider gravitation in any other point of view than ſimply as a tendency in bodies to the center of the earth, or to attend to it's effects at any conſiderable diſtance from the ſurface; it may not, however, be improper to obſerve that the operation of this principle is much more extenſive. Every portion of matter gravitates towards every other portion, in that part of the ſyſtem of nature which falls under our obſervation. The gravitation indeed of ſmall particles towards each other is inſenſible, on account of the ſuperior action of the earth;* yet it has been found, by the accurate obſervations of Dr. Maskelyne, in Scotland, that the attraction of a mountain is ſufficient to draw the plumb-line ſenſibly from the perpendicular.

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* The common experiment of two particles of the ſame kind, as

Sir I. Newton has diſcovered that the moon is retained in her orbit by the agency of a cauſe ſimilar to that by which a body falls to the ground, differing from it only in degree, and this in conſequence of the greater diſtance of the moon from the earth's center. The ſame author has demonſtrated that the planets are retained in their reſpective orbits by a principle of the ſame kind; and that the minuteſt irregularities in their motions may be ſatisfactorily deduced from the known laws of it's operation.

8. *Inactivity* may be conſidered in two lights: 1ſt. As an *inability* in matter to change it's ſtate of reſt or uniform rectilinear motion: 2d. As that quality by which it *reſiſts* any ſuch change. In this latter ſenſe it is uſually called the *force of inactivity*, the *inertia*, or the *vis inertiæ*.

The inactivity of matter, according to the former explanation, is laid down as a law of motion; the truth of which we ſhall endeavour to eſtabliſh in the next ſection.

That a body *reſiſts* any change in it's ſtate of reſt, or uniform rectilinear motion, is known from conſtant experience. We cannot move the leaſt particle of matter without ſome exertion; nor can we deſtroy any

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as oil, water, quickſilver, &c. when placed upon a ſmooth horizontal plane, running together, cannot be attributed to this cauſe. If the effect were not produced by ſome power different from gravitation, a drop of oil would run in the ſame manner towards a drop of water, which is not found to be the caſe.

any motion without perceiving ſome reſiſtance.* Thus we ſee, in general, that *inertia* is a property inherent in all bodies with which we are concerned; different indeed in different caſes, but exiſting, in a greater or leſs degree, in all. The quantity we are not at preſent conſidering; the exiſtence of the property, every one, from his own obſervation, will readily allow. Thus far indeed our common experience leads us with reſpect to the quantity of inertia, that if one portion of matter be added to another, the inertia of the whole is increaſed; and if any part be removed the inertia is leſſened. This clearly ſhews that it exiſts in every particle, and that the whole inertia is the aggregate of all it's parts.

Hence it follows, from our notion of quantity, that if to a body with a certain quantity of inertia, another, which has an equal quantity, be added, the whole inertia will be doubled; and that by the repeated addition of equal quantities, the whole inertia will be increaſed in the ſame proportion with the number of parts.

Theſe properties, which are always found to exiſt together in the ſame ſubſtance, have ſometimes been ſaid to be eſſential to matter: Whether they are, or are not *neceſſarily* united in the ſame ſubſtance it is impoſſible to decide, nor does it concern us to enquire. The buſineſs of natural philoſophy is not to find out what might have been the conſtitution of nature, but to examine what it is in fact; and to account for the phænomena, which fall under our obſervation, from thoſe properties of matter which we know by experience that it poſſeſſes.

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* It muſt be obſerved, that this reſiſtance is diſtinct from, and independent of gravity; becauſe it is perceived where gravity produces no effect; as, when a wheel is turned round it's axis, or a body moved along an horizontal plane.

9. By the *quantity of matter* in a body, we underſtand the aggregate of it's particles, each of which has a certain degree of inertia. Or, in other words, if we ſuppoſe bodies made up of particles, each of which has the ſame inertia, the quantity of matter in one, is to the quantity of matter in another, as the number of ſuch particles in the former body, to the number in the latter.*

When we conſider bodies as made up of parts, and compare them in this reſpect, it becomes neceſſary to give a definite and preciſe deſcription of thoſe parts; otherwiſe, our notion of the quantity will be vague and inaccurate. Now the only properties of matter which admit of exact compariſon, and which depend upon the number, and not upon the arrangement of the particles, are weight and inertia; either of which may properly be made uſe of as a meaſure of the *quantity* of matter; and ſince, at a given place, they are proportional to each other, as we ſhall ſhew hereafter (Art. 25), it is of little conſequence which meaſure we adopt. The inertia has been fixed upon, becauſe the gravity of a body, though invariable at the

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[*] Quantitas materiæ eſt menſura ejuſdem orta ex illius denſitate et magnitudine conjunctim. Newt. Princip. Def. 1.

Ejuſdem eſſe denſitatis dico, quarum vires inertiæ ſunt ut magnitudines. Lib. III. Prop. 6. Cor. 4.

Attendi enim oportet ad punctorum numerum, ex quibus corpus movendum eſt conſtitutum. Puncta vero ea inter ſe æqualia cenſeri debent, non quæ æque ſunt parva, ſed in quæ eadem potentia æquales exerit effectus. Eul. Mech. 139.

the ſame place, is different at different diſtances from the center of the earth; whereas, the inertia is always, and under all circumſtances, the ſame.

10. By *motion* we underſtand a change of place; and it is of two kinds, *abſolute* and *relative*.

A body is ſaid to be in *abſolute motion* when it is actually transferred from one point in fixed ſpace to another; and to be *relatively in motion*, when it's ſituation is changed with reſpect to the ſurrounding bodies.

Theſe two kinds of motion evidently coincide when the bodies, to which the reference is made, happen to be fixed. In other caſes, a body relatively in motion, or relatively at reſt, may or may not be abſolutely in motion. Thus, a ſpectator ſtanding ſtill on the ſhore, if his place be referred to a ſhip which ſails by, is relatively in motion; and the ſeveral parts of the veſſel are at reſt, with reſpect to each other, though the whole is transferred from one part of ſpace to another.

When a body always paſſes over equal parts of ſpace in equal ſucceſſive portions of time, it's motion is ſaid to be *uniform*. When the ſucceſſive portions of ſpace, deſcribed in equal times, continually increaſe, the motion is ſaid to be *accelerated*; and to be *retarded*, when thoſe ſpaces continually decreaſe. Alſo, the motion is ſaid to be *uniformly* accelerated or retarded, when the increments or decrements of the ſpaces, deſcribed in equal ſucceſſive portions of time, are always equal.

11. The *velocity* of a body, or *rate* of it's motion, is meaſured by the ſpace, *uniformly deſcribed* in a *given* time.

The given time, taken as a ſtandard, is uſually one ſecond; and the ſpace deſcribed is meaſured in feet. Thus, when v repreſents a body's velocity, v is the number of feet which the body would uniformly deſcribe in one ſecond.

If a body's motion be accelerated or retarded, the velocity at any point is not meaſured by the ſpace actually deſcribed in a given time, but by the ſpace which would have been deſcribed in the given time, if the motion had continued uniform, from that point.

12. Cor. 1. If two bodies move uniformly on the ſame line, in oppoſite directions, their relative velocity is equal to the *ſum* of their abſolute velocities, ſince the ſpace by which they uniformly approach to, or recede from, each other, in any time, is equal to the ſum of the ſpaces which they reſpectively deſcribe in that time.

When the bodies move in the ſame direction, their relative velocity is equal to the *difference* of their abſolute velocities.

13. Cor. 2. When a body moves with an uniform velocity, the ſpace deſcribed is proportional to the time of it's motion.

Let the body deſcribe a feet in one ſecond, then ſince the motion is uniform, it will deſcribe ta feet in t ſeconds; that is, the ſpace deſcribed is proportional to the time.

14. Cor. 3. When bodies have different uniform motions, the ſpaces deſcribed are proportional to the times and velocities jointly.*

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* Since the times and velocities may in each caſe be repreſented by numbers, there is no impropriety in ſpeaking of their products.

Let V and v be the velocities of two bodies A and B; T and t the times of their motions; S and s the ſpaces deſcribed. Alſo let S' be the ſpace deſcribed by B in the time T;

Then $S : S' :: V : v$ (Art. 11),
$\underline{S' : s :: T : t}$ (Art. 13),
Comp. $S : s :: TV : tv$;
that is, $S \propto TV$ (Alg. Art. 195).

Ex. Let the times be to each other as $6 : 5$, and the velocities as $2 : 3$; then $S : s :: 2 \times 6 : 3 \times 5 :: 4 : 5$.

15. Cor. 4. Since $S \propto TV$, we have $V \propto \frac{S}{T}$, and $T \propto \frac{S}{V}$, (Alg. Art. 205).

Ex. 1. Let A move uniformly through 5 feet in 3″, and B through 9 feet in 7″; required the ratio of the velocities.

$$V : v :: \frac{5}{3} : \frac{9}{7} :: 35 : 27.$$

Ex. 2. Let A's velocity be to B's velocity as 5 to 4; to compare the times in which they will deſcribe 9 and 7 feet reſpectively.

$$T : t :: \frac{9}{5} : \frac{7}{4} :: 36 : 35.$$

16. Cor.

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products. The truth of this obſervation will be evident, if the propoſition be expreſſed in different words: When the uniform velocities of two bodies are in the ratio of the numbers V and v, and the times of their motions in the ratio of the numbers T and t, the ſpaces deſcribed are in the ratio of the numbers TV and tv.

16. Cor. 5. Since the areas of rectangles are in the ratio compounded of the ratios of their ſides, if the baſes repreſent the velocities of two motions, and altitudes the times, the areas will repreſent the ſpaces deſcribed.

17. The *quantity of motion*, or *momentum*, of a body, is meaſured by the velocity and quantity of matter jointly.

Thus, if the quantities of matter in two bodies be repreſented by 6 and 7, and their velocities by 9 and 8, the ratio of 6×9 to 7×8, or 27 to 28, is *called* the ratio of their *momenta*.

18. Cor. 1. If M be the momentum of a body, Q it's quantity of matter, and V it's velocity, then ſince $M \propto QV$, we have $Q \propto \frac{M}{V}$; and $V \propto \frac{M}{Q}$.

Ex. If the quantities of motion be as 6 to 5, and the velocities as 7 to 8, what is the ratio of the quantities of matter?

Since $Q \propto \frac{M}{V}$, we have $Q : q :: \frac{6}{7} : \frac{5}{8} :: 48 : 35$.

19. Cor. 2. If M be given, $Q \propto \frac{1}{V}$; and conſequently if $Q \propto \frac{1}{V}$, M is invariable. (Algebra, Art. 206).

20. Whatever changes, or tends to change, the ſtate of reſt or uniform rectilinear motion of a body, is called *force*.

Thus, preſſure, impact, gravity, &c. are called forces.

When a force produces it's effect inſtantaneouſly, it is ſaid to be *impulſive.** When it acts inceſſantly, it is called a *conſtant* force.

Conſtant forces are of two kinds, *uniform* and *variable.* A force is ſaid to be *uniform* when it always produces *equal* effects in equal ſucceſſive portions of time; and *variable*, when the effects produced in equal times are *unequal.*

Forces which are known to us only by their effects, muſt be compared by eſtimating thoſe effects under the ſame circumſtances. Thus impulſive forces muſt be meaſured by the whole effects produced; uniform forces, by the effects produced in equal times; and variable forces, by the effects which would be produced in equal times, were they to become and continue uniform for thoſe times.

The effects produced by the actions of forces are of two kinds, velocity and momentum; and thus we have two methods of comparing them, according as we conceive them to be the cauſes of velocity or momentum.

21. The *accelerating force* is meaſured by the *velocity* uniformly generated in a given time, no regard being had to the quantity of matter moved.

Thus, if the velocities uniformly generated, in two caſes, in equal times, be as 6 to 7, the accelerating forces are *ſaid* to be in that ratio.

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* Though we cannot conceive finite effects to be produced otherwiſe than by degrees, and conſequently in ſucceſſive portions of time; yet when theſe portions are ſo ſmall as not to be diſtinguiſhable by our faculties, the effects may be ſaid to be inſtantaneous.

The accelerating force of gravity at the ſame place is invariable; for all bodies falling freely, in an exhauſted receiver, acquire equal velocities in any given time.

22. The *moving force* is meaſured by the *momentum* uniformly generated in a given time.

If the momenta thus generated, in two caſes, be as 14 to 15, the moving forces are *ſaid* to be in that ratio.

23. Cor. 1. Since the momentum is proportional to the velocity and quantity of matter, the moving force varies as the accelerating force and quantity of matter jointly.

The moving force of gravity varies as the quantity of matter moved, becauſe the accelerating force is given (Art. 21).

24. Cor. 2. Hence it follows that the accelerating force varies as the moving force directly, and the quantity of matter inverſely.

Prop. I.

25. *The vis inertiæ of any body is proportional to it's weight.*

The inertia, as was obſerved on a former occaſion, is the reſiſtance which a body makes to any change in it's ſtate of reſt or uniform motion (Art. 8); and this reſiſtance is manifeſtly the ſame in two caſes, if the ſame force, applied in the ſame manner, and for the ſame time, generate the ſame velocity.

Let two bodies *A* and *B*, equal in *weight*, be placed in two ſimilar and equal boxes, which are connected by a ſtring and hang over a fixed pulley; then theſe will exactly balance each other; and if the whole be put into motion, the gravity can neither accelerate nor retard that motion; the whole reſiſtance therefore to the communication of motion in the ſyſtem, ariſes from the inertia of the weights, the inertia of the ſtring and pulley, the friction upon the axis, and the reſiſtance of the air.*

Now let a weight *C* be added on one ſide, and let the velocity generated in any given time, in the whole ſyſtem, by this additional weight, be obſerved.

Then in the place of *A*, or *B*, ſubſtitute any other maſs of the ſame weight, and it will be found that *C* will, in the ſame time, generate the ſame velocity in this ſyſtem as in the former; and therefore, the whole reſiſtance to the communication of motion muſt be the ſame. Alſo the inertia of the ſtring and pulley, the friction of the axis, and the air's reſiſtance are the ſame in the two experiments; conſequently, the reſiſtance ariſing from the inertia of the weights is the ſame: That is, ſo long as the weight remains unaltered, whatever be the form or conſtitution of the body, the inertia is the ſame.

Alſo, ſince the whole quantity of inertia is the aggregate inertia of all the parts (Art. 8), if the *weight* be doubled, an equal quantity of *inertia* is added to the former quantity, or the whole *inertia* is doubled; and in the ſame manner, if the *weight* be increaſed in any proportion, by the repeated addition of equal *weights*, the *inertia* is increaſed in the ſame proportion.

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* This experiment may be made with great accuracy by means of a machine, invented by Mr. Atwood, for the purpoſe of examining the motions of bodies when acted upon by conſtant forces. This machine is deſcribed in his well known treatiſe on the *Rectilinear Motion and Rotation of Bodies*, (p. 299).

It may be obſerved, that the velocity generated in a given time, is the ſame, whether the ſyſtem begins to move from reſt or not; therefore the inertia is the ſame, whether the body be at reſt or in motion.

26. Cor. Since the quantity of matter is meaſured by the inertia (Art. 9), it is alſo proportional to the weight.

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SECTION II.

ON THE LAWS OF MOTION.

THE FIRST LAW.

27. *IF a body be at reſt it will continue at reſt, and if in motion it will continue to move uniformly forward in a right line, till it is acted upon by ſome external force.*

That a body at reſt cannot put itſelf in motion we know from conſtant and univerſal experience.

That a body in motion will continue to move uniformly forward in a right line till it is acted upon by ſome external force, though equally certain, is not, it muſt be allowed, equally apparent; ſince all the motions which fall under our immediate obſervation, and rectilinear motions in particular, are ſoon deſtroyed. If however we can point out the cauſes which tend to deſtroy the motions of bodies, and ſhew, experimentally, that by removing ſome of them and diminiſhing others, the motions continually become more uniform and rectilinear, we may juſtly conclude that any deviation from the firſt direction, and firſt velocity, muſt be attributed to the agency of external cauſes; and that there is no tendency in matter itſelf, either to increaſe or diminiſh any motion impreſſed upon it.

Now the cauſes which retard a body's motion, beſides colliſion, or the evident obſtruction which it meets with from ſenſible maſſes of matter, are gravity, friction, and the reſiſtance of the air; and it will appear by the following experiments that when theſe are removed, or due allowance is made for their known effects, we are neceſſarily led to infer the truth of the law above laid down.

1ſt. If a ball be thrown along a rough pavement, it's motion, on account of the many obſtacles it meets with, will be very irregular and ſoon ceaſe; but if it be bowled upon a ſmooth bowling-green, it's motion will continue longer, and be more rectilinear; and if it be thrown along a ſmooth ſheet of ice, it will preſerve both it's direction and it's motion for a ſtill longer time.

In theſe caſes, the gravity, which acts in a direction perpendicular to the plane of the horizon, neither accelerates nor retards the motion; the cauſes which produce the latter effect are colliſion, friction, and the air's reſiſtance; and in proportion as the two former of theſe are leſſened, the motion becomes more nearly uniform and rectilinear.

2d. When a wheel is accurately conſtructed, and a rotatory motion about it's axis communicated to it, if the axis and it's ſupports be well poliſhed, the motion will continue a conſiderable time; if the axis be placed upon friction wheels, the motion will continue longer; and if the apparatus be placed under the receiver of an air pump, and the air be exhauſted, the motion will continue without viſible diminution for a very long time.

In theſe inſtances, gravity, which acts equally on oppoſite points of the wheel, neither accelerates nor retards the motion; and the more care we take to remove the friction, and the reſiſtance of the air, the leſs is the firſt motion diminiſhed in a given time.

3d. If a body be projected in any direction inclined to the horizon, it deſcribes a curve which is nearly the common parabola. This effect is produced by the joint action of gravity and the motion of projection; and ſince the effect produced by the former is known, the effect produced by the latter may be determined. This, it is found, would carry the body uniformly forward in the line in which it was projected; as will fully appear when we come to the doctrine of projectiles. The deviation of the curve deſcribed from the parabolic form is ſufficiently accounted for by the reſiſtance of the air.

From theſe, and ſimilar experiments, we are led to conclude that all bodies in motion would uniformly perſevere in that motion, were they not prevented by external impediments; and that every increaſe or diminution of velocity, every deviation from the line of direction, is to be attributed to the agency of ſuch cauſes.

28. It may not be improper to obſerve, that this law ſuggeſts two methods of diſtinguiſhing between abſolute motions, and ſuch as are only apparent; one, by conſidering the cauſes which produce the motions; and the other, by attending to the effects with which the motions are accompanied.[*] 1ſt. We

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[*] Newt. Princip. Schol. ad Def.

1ſt. We may ſometimes diſtinguiſh abſolute motion, or change of abſolute motion, from that which is merely apparent, by conſidering the cauſes which produce them.

When two bodies are abſolutely at reſt, they are relatively ſo; and the appearance is the ſame, when they are moving in the ſame direction, at the ſame rate; a relative motion therefore can only ariſe from an abſolute motion, or change of abſolute motion, in one or both of the bodies. We have ſeen alſo, in the laſt article, that motion, or change of motion cannot be produced but by force impreſſed; and therefore, if we know that ſuch a cauſe exiſts, and acts upon one of the bodies, and not upon the other, we conclude that the relative motion ariſes from a change in the ſtate of reſt, or abſolute motion of the former; and that with reſpect to the latter, the effect is merely apparent. Thus, when a perſon on ſhipboard obſerves the coaſt receding from him, he is convinced that the appearance ariſes from a motion, or change of motion, in the ſhip, upon which a cauſe, ſufficient to produce this effect, acts, namely, the force of the wind or tide.

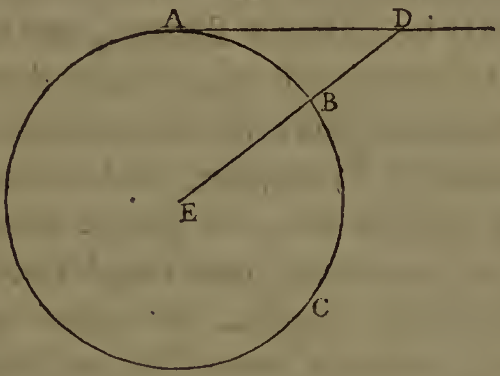
The preceſſion of the equinoxes ariſes from a real motion in the earth, and not from any motion in the heavenly bodies; becauſe we know that there is a force impreſſed upon the earth, which is ſufficient to account for the appearance.

2d. Abſolute motion may ſometimes be diſtinguiſhed from apparent motion, by the effects produced.

If a body be abſolutely in motion, it endeavours by it's inactivity to proceed in the line of it's direction; if the motion be only apparent, there is no ſuch tendency.

It is in conſequence of the tendency to perſevere in rectilinear motion that a body revolving in a circle conſtantly endeavours to recede from the center. The effort, thus produced, is called a *centrifugal force*; and as it ariſes from abſolute motion only, whenever it is obſerved, we are convinced that the motion is real.

In order to ſee the nature and origin of this force, ſuppoſe a body to deſcribe the circle *ABC*; then at any point *A*, it is moving in the direction



of the tangent *AD*, and in this direction, by the firſt law of motion, it endeavours to proceed; alſo, ſince every point *D* in the tangent is without the circle, this tendency to move on in the direction of the tangent, is a tendency to recede from the center of motion; and the body will actually fly off, unleſs it is prevented by an adequate force.

The following experiment is given by SIR I. NEWTON to ſhew the effect of the centrifugal force, and to prove that it always accompanies an abſolute circular motion.

Let a bucket, partly filled with water, be ſuſpended by a ſtring, and turned round till the ſtring is conſiderably twiſted; then let the ſtring be ſuffered to untwiſt itſelf, and thus communicate a circular motion to the veſſel. At firſt the water remains at reſt, and it's ſurface is ſmooth and undiſturbed; but as it gradually acquires the motion of the bucket, the ſurface grows concave towards the center, and the water aſcends up the ſides, thus endeavouring to recede from the axis of motion; and this effect is obſerved gradually to increaſe with the abſolute velocity of the water, till at length the water and the bucket are relatively at reſt. When this is the caſe, let the bucket be ſuddenly ſtopped, and the abſolute motion of the water will be gradually diminiſhed by the friction of the veſſel; the concavity of the ſurface is alſo diminiſhed by degrees, and at length, when the water is again at reſt, the ſurface becomes plane. Thus we find that the centrifugal force does not depend upon the relative, but upon the abſolute motion, with which it always begins, increaſes, decreaſes, and diſappears.

A ſingle inſtance will be ſufficient to ſhew the great utility of this concluſion in natural philoſophy.

The diurnal rotation of the heavenly bodies may, as far as the appearance is concerned, be accounted for, either by ſuppoſing the heavens to revolve from eaſt to weſt, and complete a revolution in twenty-four hours; or, the earth to revolve from weſt to eaſt, in the ſame time: but the ſenſible diminution of gravity as we proceed towards the equator, and the oblate figure of the earth, which are the effects of a centrifugal force, prove that the appearance is to be aſcribed to a real motion in the earth.

THE SECOND LAW OF MOTION.

29. *Motion, or change of motion, is proportional to the force impreſſed, and takes place in the direction in which the force acts.*

It has been ſeen in the preceding articles, that no motion or change of motion is ever produced in a body without *ſome* force impreſſed; we now aſſert that it cannot be produced without an *adequate* force; that when bodies act upon each other, the effects are not variable and accidental, but ſubject to general laws. Thus, whatever happens in one inſtance, will, under the ſame circumſtances, happen again; and when any alteration takes place in the cauſe, there will be a correſponding and proportional alteration in the effect produced. Were not cauſe and effect thus connected with, and related to, each other, we could not pretend to lay down any general rules reſpecting the mutual actions of bodies; experiment could only furniſh us with detached and iſolated facts, wholly inapplicable on other occaſions; and that harmony, which we cannot but obſerve and admire in the material world, would be loſt.

In order to underſtand the meaning and extent of this law of motion, it will be convenient to divide it into two caſes; and to point out ſuch facts, under each head, as tend to eſtabliſh it's truth.

1ſt. The ſame force, acting freely for a given time, will always produce the ſame effect, in the direction in which it acts.

Ex. 1. If a body, in one inſtance, fall perpendicularly through $16\frac{1}{12}$ feet in a ſecond, and thus acquire a velocity which would carry it, uniformly, through $32\frac{1}{6}$ feet in that time, it will always, under the ſame circumſtances, acquire the ſame velocity.

The effect produced is the ſame, whether the body begins to move from reſt or not.

Ex. 2. If a body be projected perpendicularly downwards, the velocity of projection, meaſured in feet (Art. 11), will, in one ſecond, be increaſed by $32\frac{1}{6}$; and if it be projected upwards, it will, in one ſecond, be diminiſhed by that quantity.

Ex. 3. If a body be projected obliquely, gravity will ſtill produce it's effect in a direction perpendicular to the horizon; and the body, which by it's inactivity, would have moved uniformly forward in the line of it's firſt motion, will, at the end of one ſecond, be found $16\frac{1}{12}$ feet below that line; having thus acquired a velocity of $32\frac{1}{6}$ feet per ſecond, in the direction of gravity.

2d. If the force impreſſed be increaſed or diminiſhed in any proportion, the motion communicated will be increaſed, or diminiſhed in the ſame proportion.

Ex. If a body deſcend along an inclined plane, the length of which is twice as great as it's height, the force which accelerates it's motion is half as great as the force of gravity; and, allowing for the effect of friction, and the reſiſtance of the air, the velocity generated in any time is half as great as it would have been, had the body fallen, for the ſame time, by the whole force of gravity.*

30. In eſtimating the effect of any force, two circumſtances are to be attended to; firſt, we muſt conſider what force is actually impreſſed; for this alone can produce a change in the ſtate of motion or quieſcence of a body. Thus, the effect of a ſtream upon the floats of a water-wheel is not produced by the whole force of the ſtream, but by that part of it which ariſes from the exceſs of the velocity of the water above that of the wheel; and it is nothing, if they move with equal velocities. Secondly, we muſt

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* The experiments which moſt ſatisfactorily prove the truth of this law of motion, are made with Mr. Atwood's machine, mentioned on a former occaſion (Art. 25).

Let two weights, each of which is repreſented by $9m$, balance each other on this machine; and obſerve what velocity is generated, in one ſecond, when a weight $2m$ is added to either of them. Again, let the weights $8m$, $8m$, be ſuſtained, as before, and add $4m$ to one of them, then the velocity generated in one ſecond is twice as great as in the former inſtance; ſince, therefore, the maſs to be moved is the ſame in both caſes, viz. $20m$ together with the inertia of the machine, it is manifeſt that when the moving force is doubled (Art. 23), the momentum generated is alſo doubled; and, by altering the ratio of the weights, it may be ſhewn, in any other caſe, that the momentum communicated is proportional to the moving force impreſſed.

conſider in what direction the force acts; and take that part of it, only, which lies in the direction in which we are eſtimating the effect. Thus, the force of the wind actually impreſſed upon the ſails of a windmill, is not wholly employed in producing the circular motion; and therefore in calculating it's effect, in this reſpect, we muſt determine what part of the whole force acts in the direction of the motion.

In the following pages, we ſhall ſee a great variety of inſtances in which this method of eſtimating the effects of forces is applied; and the concluſions thus deduced, being found, without exception, to agree with experiment, we cannot but admit the truth of the principle.

31. Cor. Since the effect produced upon each other by two bodies, depends upon their relative velocity, it will always be the ſame whilſt this remains unaltered, whatever be their abſolute motions.

THE THIRD LAW OF MOTION.

32. *Action and reaction are equal, and in oppoſite directions.*

Matter not only perſeveres in it's ſtate of reſt or uniform rectilinear motion, but alſo by it's inertia reſiſts any change. Our experience with reſpect to this reaction, or oppoſition to the force impreſſed, is ſo conſtant and univerſal, that the very ſuppoſition of it's non-exiſtence appears to be abſurd. For who can conceive a preſſure without ſome ſupport of that preſſure? Who can ſuppoſe a weight to be raiſed without force or exertion? Thus far then we are aſſured by our ſenſes, that whenever one body acts upon another, there is *ſome* reaction: The law farther aſſerts, that the reaction is *equal in quantity* to the action.

By *action*, we here underſtand moving force, which according to the definition (Art. 22), is meaſured by the momentum which is, or would be generated, in a given time; and to determine whether action and reaction, in this ſenſe of the words, are equal or not, recourſe muſt be had to experiment.

Take two ſimilar and equal cylindrical pieces of wood, in one of which is fixed a ſmall ſteel point; ſuſpend them by equal ſtrings, and let one of them deſcend through any arc and impinge upon the other at reſt; then, by means of the ſteel point, the two bodies will move on together as one maſs, and with a velocity equal to half the velocity of the impinging body. Thus the momentum, which is meaſured by the quantity of matter and velocity taken jointly, remains unaltered; or, as much momentum as is gained by the body ſtruck, ſo much is taken from the momentum of the ſtriking body, or communicated to it in the oppoſite direction.

If the ſtriking body be loaded with lead, and thus made twice as heavy as the other, the common velocity after impact is found to be to the velocity of the impinging body :: 2 : 3; and becauſe the joint maſs after impact : quantity of matter in the ſtriking body :: 3 : 2, the momentum after impact : momentum before :: 3 × 2 : 2 × 3, or in a ratio of equality, as in the former caſe.

In making experiments to eſtabliſh this third law of motion, allowance muſt be made for the air's reſiſtance; and care muſt be taken to obtain a proper meaſure of the velocity before and after impact. See Sir I. Newton's Scholium to the Laws of Motion.

33. The third law of motion is not confined to caſes of actual impact; the effects of preſſures and attractions, in oppoſite directions, are alſo equal.

When two bodies ſuſtain each other, the preſſures in oppoſite directions muſt be equal, otherwiſe motion would enſue; and if motion be produced by the exceſs of preſſure on one ſide, the caſe coincides with that of impact.*

When one body attracts another, it is itſelf alſo equally attracted; and as much momentum as is thus communicated to one body, will alſo be communicated to the other in the oppoſite direction.

A loadſtone and a piece of iron, equal in weight, and floating upon ſimilar and equal pieces of cork, approach each other with equal velocities, and therefore with equal momenta; and when they meet, or are

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* The effects of preſſure and impact are manifeſtly of the ſame kind, and produced in the ſame way; exceſs of preſſure, on one ſide, produces momentum, and equal and oppoſite momenta ſupport each other by oppoſite preſſures.

Thus alſo preſſures may be compared, either by comparing the weights which they ſuſtain, or by the momenta which they would generate under the ſame circumſtances.

are kept aſunder by any obſtacle, they ſuſtain each other by equal and oppoſite preſſures.

34. Cor. Since the action and reaction are equal at every inſtant of time, the whole effect of the action in a finite time, however it may vary, is equal to the effect of the reaction; ſince the whole effects are made up of the effects produced in every inſtant.

SCHOLIUM.

35. Theſe laws are the ſimpleſt principles to which motion can be reduced, and upon them the whole theory depends. They are not indeed ſelf-evident, nor do they admit of accurate proof by experiment, on account of the great nicety required in making the experiments, of the effects of friction, and of the air's reſiſtance, which cannot entirely be removed. They are however conſtantly, and invariably, ſuggeſted to our ſenſes, and they agree with experiment as far as experiment can go; and the more accurately the experiments are made, and the greater care we take to remove all thoſe impediments which tend to render the concluſions erroneous, the more nearly do the experiments coincide with theſe laws.

Their truth is alſo eſtabliſhed upon a different ground; from theſe *general* principles innumerable *particular* concluſions have been deduced; ſometimes the deductions are ſimple and immediate, ſometimes they are made by tedious and intricate operations; yet they are all, without exception, conſiſtent with each other and with experiment: it follows therefore that the principles, upon which the calculations are founded, are true.*

36. It will be neceſſary to remember, that the laws of motion relate, *immediately*, to the actions of particles of matter upon each other, or to thoſe caſes in which the whole maſs may be conceived to be collected in a point; not to *all* the effects that may *eventually* be produced in the ſeveral particles of a ſyſtem.

A body may have a rectilinear and rotatory motion given it at the ſame time, and it will retain both. The action alſo, or reaction, may be applied at a mechanical advantage or diſadvantage, and thus produce, upon the whole, very different momenta; theſe effects depend upon principles which are not here conſidered; but which muſt be attended to in computing ſuch effects.

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* Atwood on the Motions of Bodies. p. 358.

SECTION III.

ON THE COMPOSITION AND RESOLUTION OF MOTION.

Prop. II.

37. *TWO lines, which repreſent the momenta communicated to the ſame or equal bodies, will repreſent the ſpaces uniformly deſcribed by them in equal times; and converſely, the lines which repreſent the ſpaces uniformly deſcribed by them in equal times, will repreſent their momenta.*

The momenta of bodies may be repreſented by numbers, as was ſeen Art. 17; but in many caſes it will be much more convenient to repreſent them by lines, becauſe lines will expreſs not only the quantities of the momenta, but alſo the directions in which they are communicated.

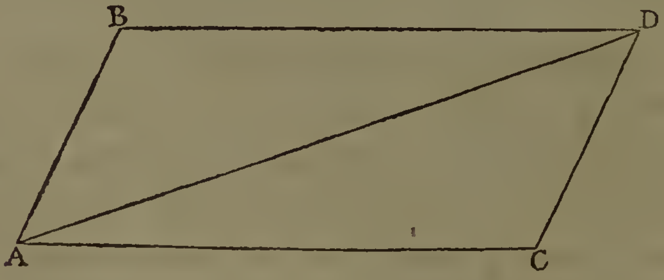
Any line drawn in the proper direction, may be taken to repreſent one momentum; but to repreſent a ſecond, a line, in the direction of the latter motion, muſt be taken in the ſame proportion to the former line, that the ſecond momentum has to the firſt.

Let two lines, thus taken, repreſent the momenta communicated to the ſame, or equal bodies; then ſince $M \propto V \times Q$ (Art. 17), and Q is here given, $M \propto V$; therefore the lines which repreſent the momenta, will alſo repreſent the velocities, or the ſpaces uniformly deſcribed in equal times. Again, if the lines repreſent the ſpaces uniformly deſcribed in equal times, they repreſent the velocities, and ſince Q is given, $V \propto QV \propto M$; therefore the lines repreſent the momenta.

Prop. III.

38. *Two uniform motions, which, when communicated ſeparately to a body, would cauſe it to deſcribe the adjacent ſides of a parallelogram in a given time, will, when they are communicated at the ſame inſtant, cauſe it to deſcribe the diagonal in that time; and the motion in the diagonal will be uniform.*

Let a motion be communicated to a body at A, which would cauſe it to move uniformly from A to B in T'', and at the ſame inſtant, another motion which alone would cauſe it to move uniformly from A to C in T''; complete the parallelogram BC, and draw the diagonal



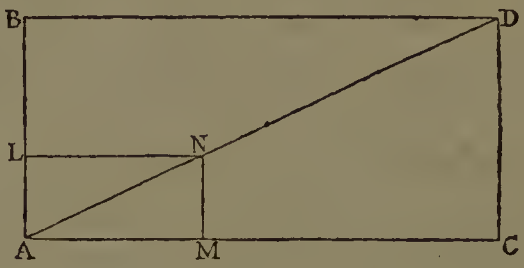
diagonal *AD*; then the body will arrive at the point *D*, in *T*″, having deſcribed *AD* with an uniform motion.

For the motion in the direction *AC* can neither accelerate nor retard the approach of the body to the line *BD* which is parallel to *AC*, (Art. 29. Ex. 3); hence the body will arrive at *BD*, in the ſame time that it would have done, had no motion been communicated to it in the direction *AC*, that is, in *T*″. In the ſame manner, the motion in the direction *AB* can neither make the body approach to, nor recede from, *CD*; therefore, in conſequence of the motion in the direction *AC*, it will arrive at *CD* in the ſame time that it would have done, had no motion been communicated in the direction *AB*, that is in *T*″. Hence it follows that, in conſequence of the two motions, the body will be found both in *BD* and *CD* at the end of *T*″, and will therefore be found in *D*, the point of their interſection.

Alſo, ſince a body in motion continues to move uniformly forward in a right line, till it is acted upon by ſome external force (Art. 27), the body *A* muſt have deſcribed the right line *AD*, with an uniform motion.

39. To illuſtrate this propoſition, ſuppoſe a plane *ABDC*, as the deck of a ſhip, to be carried uniformly forward, and let the point *A* deſcribe the line *AC* in *T*″; alſo, let a body move uniformly in this plane from *A* to *B*, in the ſame time. Complete the parallelogram *BC*, and draw the diagonal *AD*. Then at the end of *T*″ the body, by it's own motion, will arrive at *B*; alſo by the motion of the plane, *AB* will be brought in to the ſituation *CD*, and the point *B*

will coincide with D; therefore the body will upon the whole, at the end of T'', be found in D. In any other time t'', let the point A be carried from A to



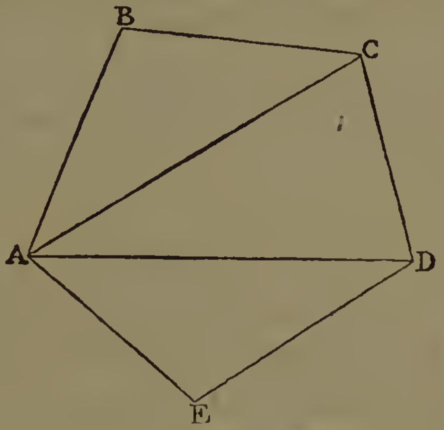
M by the motion of the plane, and the body from A to L by it's own motion; complete the parallelogram $ALNM$, and join AN; then, as in the preceding caſe, the body will, at the end of t'', be found in N; and ſince the motions in the directions AC, AB are uniform, $T : t :: AC : AM :: AB : AL$ (Art. 13); that is, the ſides of the parallelograms, about the common angle LAM, are proportional, and conſequently the parallelograms are about the ſame diagonal AD (Euc. 26.6); therefore the body at the end of any time t'' will be found in the diagonal AD. It will alſo move uniformly in the diagonal; for, from the ſimilar triangles AMN, ACD, we have $AD : AN :: AC : AM :: T : t$, or the ſpaces deſcribed are proportional to the times. (Vid. Art. 10).

40. Cor. 1. The ſame reaſoning is applicable to the motion of a point.

41. Cor. 2. If two ſides of a triangle, AB, BD, taken in order, repreſent the ſpaces over which two uniform motions would, ſeparately, carry a body in a given time; when theſe motions are communicated at the ſame inſtant to the body at A, it will deſcribe the third ſide AD, uniformly, in that time.

For, if the parallelogram *BC* be completed, the ſame motion, which would carry a body uniformly from *B* to *D*, would, if communicated at *A*, carry it in the ſame manner from *A* to *C*; and in conſequence of this motion, and of the motion in the direction *AB*, the body would uniformly deſcribe the diagonal *AD*, which is the third ſide of the triangle *ABD*.

42. Cor. 3. In the ſame manner, if the lines *AB*, *BC*, *CD*, *DE*, taken in order, repreſent the ſpaces over which any uniform motions would, ſeparately, carry a body, in a given time, when theſe motions are communicated at the ſame inſtant, the body will deſcribe the line *AE*, which completes the figure, in that time; and the motion in this line will be uniform.



43. Cor. 4. If *AD* repreſent the uniform velocity of a body, and any parallelogram *ABDC* (Art. 38) be deſcribed about it, the velocity *AD* may be ſuppoſed to ariſe from the two uniform velocities *AB*, *AC*, or *AB*, *BD*; and if one of them, *AB*, be by any means taken away, the velocity remaining will be repreſented by *AC* or *BD*.

44. Def. A force is ſaid to be *equivalent* to any

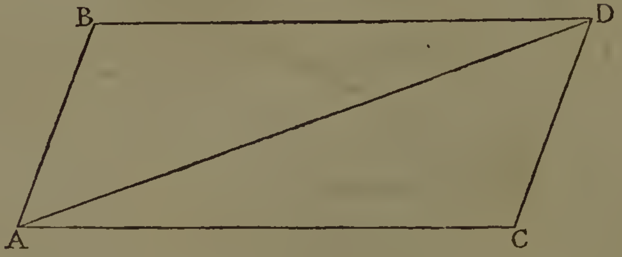
number

number of forces, when it will, *ſingly*, produce the ſame effect that the others produce *jointly*, in any given time.

PROP. IV.

45. *If the adjacent ſides of a parallelogram repreſent the quantities and directions of two forces, acting at the ſame time upon a body, the diagonal will repreſent one equivalent to them both.*

Let AB, AC repreſent two forces acting upon a body at A, then they repreſent the momenta communicated to it in thoſe directions (Art. 22), and conſequently the ſpaces which it would uniformly deſcribe in equal times (Art. 37). Complete the parallelogram CB, and draw the diagonal AD; then, by the laſt propoſition, AD is the ſpace uniformly deſcribed in the ſame time, when the two motions are communicated to the body at the ſame inſtant; and ſince AB, AC and AD, repreſent the ſpaces uniformly deſcribed by the ſame body, in equal times, they repreſent the momenta, and therefore the forces acting in thoſe directions; that is, the forces AB, AC,* acting at the ſame time, produce a force which is repreſented, in quantity and direction, by AD.



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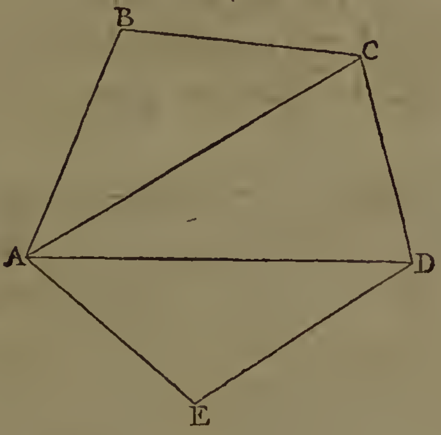
* In this, and many other caſes, where forces are repreſented by lines, the lines are uſed, for the ſake of brevity, to expreſs the forces which they repreſent.

Def. The force repreſented by *AD* is ſaid to be *compounded* of the two *AB*, *AC*.

46. Cor. 1. If two ſides of a triangle, taken in order, repreſent the quantities and directions of two forces, the third ſide will repreſent a force equivalent to them both.

For a force repreſented by *BD*, acting at *A*, will produce the ſame effect that the force *AC*, which is equal to it and in the ſame direction, will produce; and *AB*, *AC*, are equivalent to *AD*; therefore *AB*, *BD* are alſo equivalent to *AD*.

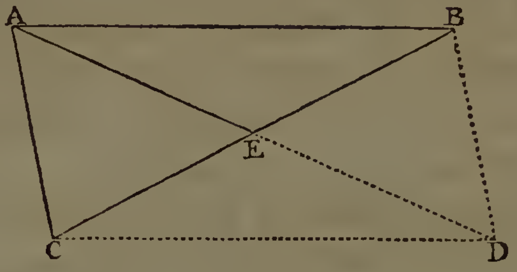
47. Cor. 2. If any lines *AB*, *BC*, *CD*, *DE*, taken in order, repreſent the quantities and directions of forces communicated at the ſame time to a body at *A*, the line *AE*, which completes the figure, will repreſent a force equivalent to them all.



For the two *AB*, *BC* are equivalent to *AC*; alſo, *AC*, *CD*, that is, *AB*, *BC*, *CD*, are equivalent to *AD*; in the ſame manner *AD*, *DE*, that is, *AB*, *BC*, *CD* and *DE*, are equivalent to *AE*.

48. Cor. 3. Let *AB* and *AC* repreſent the quantities and directions of two forces, join *BC* and draw

AE biſecting it in *E*, then will 2 *AE* repreſent a force

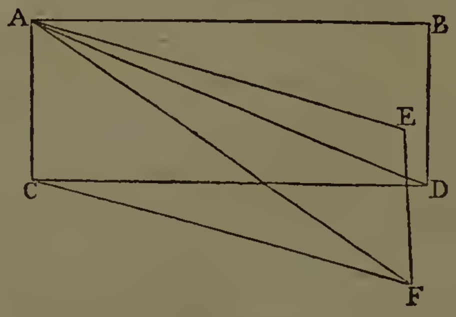


equivalent to them both.

For, if the parallelogram be completed, ſince the diagonals biſect each other, *AD*, which repreſents a force equivalent to *AB* and *AC*, is equal to 2 *AE*.

49. Cor. 4. If the angle at which two given forces act be diminiſhed, the compound force is increaſed.

Let *AB*, *AC* be the two given forces, complete



the parallelogram *ABDC* and draw the diagonal *AD*, this repreſents the compound force. In the ſame manner, if *AE* be taken equal to *AB*, and *AE*, *AC*, repreſent the two forces, then *AF*, the diagonal of the parallelogram *AEFC*, repreſents the compound force; and ſince the angle *BAC* is greater than the angle *EAC*, *ACD* which is the ſupplement of the former, is leſs than *ACF* the ſupplement of the latter; alſo, *CF* = *AE*

$AE = AB = CD$; therefore in the two triangles ACD, ACF, the ſides AC, CD are reſpectively equal to AC, CF, and the $\angle ACD$ is leſs than the $\angle ACF$; conſequently AD is leſs than AF (Euc. 24. 1).

50. Cor. 5. Two given forces produce the greateſt effect when they act in the ſame direction, and the leaſt when they act in oppoſite directions; for, in the former caſe, the diagonal AF becomes equal to the ſum of the ſides AC, CF; and in the latter, to their difference.

51. Cor. 6. Two forces cannot keep a body at reſt, unleſs they are equal and in oppoſite directions.

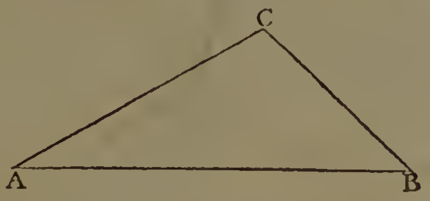
For this is the only caſe in which the diagonal, repreſenting the compound force, vaniſhes.

52. Cor. 7. In the compoſition of forces, force is loſt; for the forces repreſented by the two ſides AB, BD (Art. 45), by compoſition produce the force repreſented by AD; and the two ſides AB, BD, of a triangle, are greater than the third ſide AD.

Prop. V.

53. *If a body, at reſt, be acted upon at the ſame time by three forces which are repreſented in quantity and direction by the three ſides of a triangle, taken in order, it will remain at reſt.*

Let AB, BC, and CA, repreſent the quantities and



directions of three forces acting at the ſame time upon a body a body at *A*; then ſince *AB* and *BC* are equivalent to *AC* (Art. 46), *AB*, *BC* and *CA*, are equivalent to *AC* and *CA*; but *AC* and *CA*, which are equal and in oppoſite directions, keep the body at reſt; therefore *AB*, *BC*, and *CA*, will alſo keep the body at reſt.

Prop. VI.

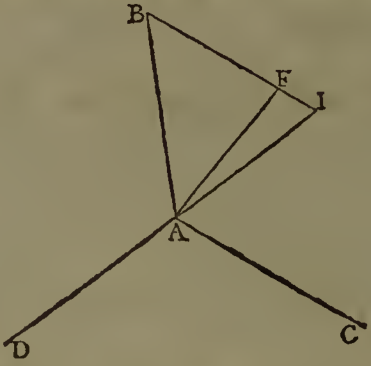
54. *If a body be kept at reſt by three forces, and two of them be repreſented in quantity and direction by two ſides AB, BC,* of a triangle, the third ſide, taken in order, will repreſent the quantity and direction of the other force.*

Since *AB*, *BC* repreſent the quantities and directions of two of the forces, and *AB*, *BC* are equivalent to *AC*, the third force muſt be ſuſtained by *AC*; therefore *CA* muſt repreſent the quantity and direction of the third force (Art. 51).

Prop. VII.

55. *If a body be kept at reſt by three forces, acting upon it at the ſame time, any three lines, which are in the directions of theſe forces, and form a triangle, will repreſent them.*

Let three forces acting in the directions *AB*, *AC*, *AD*, keep the body *A* at reſt. In *AB* take any point *B*,



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* Fig. Art. 53.

B, and through *B* draw *BI* parallel to *AC*, meeting *DA* produced in *I*; then will *AB*, *BI* and *IA* repreſent the three forces.

For *AB* being taken to repreſent the force in that direction, if *BI* do not repreſent the force in the direction *AC* or *BI*, let *BF* be taken to repreſent it; join *AF*; then ſince three forces keep the body at reſt, and *AB*, *BF* repreſent the quantities and directions of two of them, *FA* will repreſent the third (Art. 54), that is, *FA* is in the direction *AD*, which is impoſſible (Euc. 11. 1. Cor.); therefore *BI* repreſents the force in the direction *AC*; and conſequently *IA* repreſents the third force (Art. 54).

56. Cor. 1. If three forces keep a body at reſt, they act in the ſame plane; becauſe the three ſides of a triangle are in the ſame plane (Euc. 2. 11).

57. Cor. 2. If a body be kept at reſt by three forces, any two of them are to each other inverſely as the ſines of the angles which the lines of their directions make with the direction of the third force.

Let *ABI* be a triangle whoſe ſides are in the directions of the forces; then theſe ſides repreſent the forces; and *AB* : *BI* :: ſin. *BIA* : ſin. *BAI* :: ſin. *IAC* : ſin. *BAI* :: ſin. *CAD* : ſin. *BAD*.

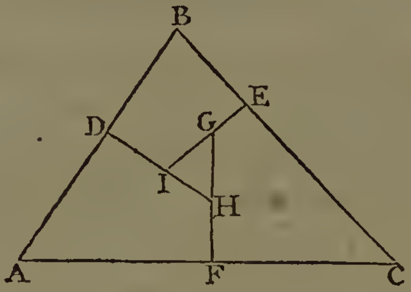
58. Cor. 3. If a body, at reſt, be acted upon at the ſame time by three forces, in the directions of the ſides of a triangle taken in order, and any two of them be to each other inverſely as the ſines of the angles which their directions make with the direction of the third, the body will remain at reſt.

For in this caſe the forces will be proportional to the three ſides of the triangle, and conſequently they will ſuſtain each other (Art. 53).

Prop. VIII.

59. *If a body be kept at reſt by three forces, and lines be drawn at right angles to the directions in which they act, forming a triangle, the ſides of this triangle will repreſent the quantities of the forces.*

Let *AB*, *BC*, *CA* be the directions in which the forces act; then the lines *AB*, *BC*, *CA* will repreſent the forces (Art. 55). Draw the perpendiculars *DH*, *EI*, *FG*, forming a triangle *GHI*; then ſince the four angles of the quadrilateral figure *ADHF* are equal to four right angles, and the angles at *D* and *F* are right angles, the remaining angles *DHF*, *DAF* are equal to two right angles, or to the two angles *DHF*, *DHG*; conſequently, the angle *DAF* is equal to the angle *IHG*. In the ſame manner it may be ſhewn, that the angles *ABC*, *BCA* are reſpectively equal to *GIH*, *HGI*; therefore the triangles *ABC* and *GHI* are equiangular; hence, the ſides about their equal angles being proportional, the forces which are proportional to the lines *AB*, *BC*, *CA*, are proportional to the lines *HI*, *IG*, *GH*.



Cor. If the lines *DH*, *EI*, *FG* be equally inclined to the lines *DB*, *EC*, *FA*, the ſides of the triangle *GHI* will repreſent the forces.

PROP. IX.

60. *If any number of forces, repreſented in quantity and direction by the ſides of a polygon, taken in order, act at the ſame time upon a body at reſt, they will keep it at reſt.*

Let *AB*, *BC*, *CD*, *DE* and *EA* (Fig. Art. 47), repreſent the forces; then ſince *AB*, *BC*, *CD* and *DE* are equivalent to *AE* (Art. 47), *AB*, *BC*, *CD*, *DE*, and *EA*, are equivalent to *AE* and *EA*; that is, they will keep the body at reſt.

PROP. X.

61. *If any number of lines, taken in order, repreſent the quantities and directions of forces which keep a body at reſt, theſe lines will form a polygon.*

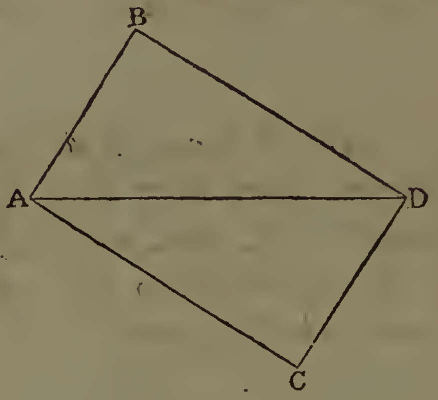
Let *AB*, *BC*, *CD* and *DE* repreſent forces which keep a body at reſt (Fig. Art. 47); then the point *E* coincides with *A*. If not, join *AE*; then *AB*, *BC*, *CD* and *DE* are equivalent to *AE*; and the body would be put in motion by a ſingle force *AE*, which is contrary to the ſuppoſition; therefore the point *E* coincides with *A*, and the lines form a polygon.

This and the laſt propoſition are true when the forces act in different planes.

PROP. XI.

62. *A ſingle force may be reſolved into any number of forces.*

Since the ſingle force *AD* is equivalent to the two *AB*, *BD*, it may be conceived to be made up of, or reſolved into, the two *AB*, *BD*. The force *AD* may therefore be reſolved into as many pairs of forces as there can be triangles deſcribed upon *AD*, or parallelograms about it. Alſo *AB*, or *BD*, may be reſolved into two; and, by proceeding in the ſame manner, the original force may be reſolved into any number of others.



63. Cor. 1. If two forces are together equivalent to *AD*, and *AB* be one of them, *BD* is the other.

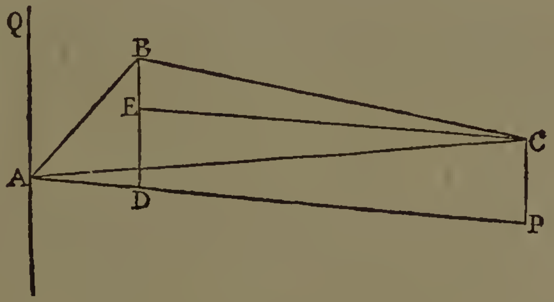
64. Cor. 2. If the force *AD* be reſolved into the two *AB*, *BD*, and *AB* be wholly loſt, or deſtroyed, the effective part of *AD* is repreſented in quantity and direction by *BD*.

65. Cor. 3. In the reſolution of forces, the whole quantity of force is increaſed. For the force repreſented by *AD* is reſolved into the two *AB*, *BD*, which are together greater than *AD* (Euc. 20. 1).

PROP. XII.

66. *The effects of forces, when eſtimated in given directions, are not altered by compoſition or reſolution.*

Let two forces AB, BC, and the force AC which is equivalent to them both, be eſtimated in the directions



AP, AQ. Draw BD, CP parallel to AQ; and CE parallel to AP. Then the force AB is equivalent to the two AD, DB; of which AD is in the direction AP, and DB in the direction AQ; in the ſame manner BC is equivalent to the two BE, EC; the former of which is in the direction BD or QA, and the latter in the direction EC or AP: therefore the forces AB, BC, when eſtimated in the directions AP, AQ, are equivalent to AD, EC, DB and BE; or, AD, DP, DB and BE, becauſe EC is equal to DP; and ſince DB and BE are in oppoſite directions, the part EB of the force DB is deſtroyed by BE; conſequently, the forces are equivalent to AP, DE, or AP, PC. Alſo AC, when eſtimated in the propoſed directions, is equivalent to AP, PC; therefore the effective forces in the directions AP, AQ are the ſame, whether we eſtimate *AB* and *BC*, in thoſe directions, or *AC*, which is equivalent to them.

67. Cor. When *AP* coincides with *AC*, *EC* alſo coincides with it, and *D* coincides with *E*. In this caſe the forces *DB*, *BE* wholly deſtroy each other; and thus, in the compoſition of forces, force is loſt.

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SECTION IV.

ON THE MECHANICAL POWERS.

68. THE mechanical powers are the moſt ſimple inſtruments uſed for the purpoſe of ſupporting weights, or communicating motion to bodies, and by the combination of which, all machines, however complicated, are conſtructed.

Theſe powers are ſix in number, viz. the *lever*; the *wheel and axle*; the *pulley*; the *inclined* plane; the *wedge*; and the *ſcrew.*

Before we enter upon a particular deſcription of theſe inſtruments and the calculation of their effects, it is neceſſary to premiſe, that when any forces are applied to them, they are themſelves ſuppoſed to be at reſt; and conſequently, that they are either without weight, or that the parts are ſo adjuſted as to ſuſtain each other. They are alſo ſuppoſed to be perfectly ſmooth; no allowance being made for the effects of friction.

When two forces act upon each other by means of any machine, one of them is, for the ſake of diſtinction, called the *power* and the other the *weight.*

ON THE LEVER.

69. Def. The *Lever* is an inflexible rod, moveable upon a point which is called *the fulcrum*, or *center of motion.*

The power and weight are ſuppoſed to act in the plane in which the lever is moveable round the fulcrum, and tend to turn it in oppoſite directions.

70. The properties of the lever cannot be deduced immediately from the propoſitions laid down in the laſt ſection, becauſe the forces acting upon the lever are not applied at a point, which is always ſuppoſed to be the caſe in the compoſition and reſolution of forces; they may however be derived from the following principles, the truth of which will readily be admitted.

Ax. 1. *If two weights balance each other upon a ſtraight lever, the preſſure upon the fulcrum is equal to the ſum of the weights, whatever be the length of the lever.* *

Ax. 2. *If a weight be ſupported upon a lever which reſts on two fulcrums, the preſſure upon the fulcrums is equal to the whole weight.*

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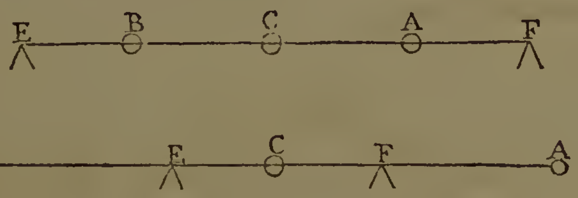
* The effect produced by the gravity of the lever is not taken into conſideration, unleſs it be expreſsly mentioned.

Ax. 3. *Equal forces, acting perpendicularly at the extremities of equal arms of a lever, exert the ſame effort to turn the lever round.*

Prop. XIII.

71. *If two equal weights, act perpendicularly upon a ſtraight lever, the effort to put it in motion, round any fulcrum, will be the ſame as if they acted together at the middle point between them.*

Let *A* and *B* be two equal weights, acting perpendicularly upon the lever *FB*, whoſe fulcrum is *F*.



Biſect *AB* in *C*; make $CE = CF$; and at *E* ſuppoſe another fulcrum to be placed.

Then ſince the two weights *A* and *B* are ſupported by *E* and *F*, and theſe fulcrums are ſimilarly ſituated with reſpect to the weights, each ſuſtains an equal preſſure; and therefore the weight ſuſtained by *E* is equal to half the ſum of the weights. Now let the weights *A* and *B* be placed at *C*, the middle point between *A* and *B*, and conſequently the middle point between *E* and *F*; then ſince *E* and *F* ſupport the whole weight *C*, and are ſimilarly ſituated with reſpect to it, the fulcrum *E* ſupports half the weight; that is, the preſſure upon *E* is the ſame whether the weights are placed at *A* and *B*, *B*, or collected in *C*, the middle point between them; and therefore, the effort to put the lever in motion round *F*, is the ſame on either ſuppoſition.

72. Cor. If a weight be formed into a cylinder *AB* (Fig. Art. 73) which is every where of the ſame denſity, and placed parallel to the horizon, the effort of any part *AD*, to put the whole in motion round *C*, is the ſame as if this part were collected at *E*, the middle point of *AD*.

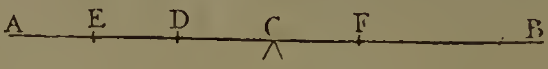
For the weight *AD* may be ſuppoſed to conſiſt of pairs of equal weights, equally diſtant from the middle point.

What is here affirmed of weights, is true of any forces which are proportional to the weights, and act in the ſame directions.

Prop. XIV.

73. *Two weights, or two forces, acting perpendicularly upon a ſtraight lever, will balance each other, when they are reciprocally proportional to their diſtances from the fulcrum.*

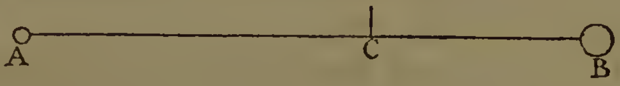
Caſe 1. When the weights act on *contrary* ſides of the fulcrum. Let x and y be the two weights, and let them be formed into the cylinder *AB*, which is every where of the ſame denſity. Biſect *AB* in *C*; then this cylinder



will balance itſelf upon the fulcrum *C* (Art. 72). Divide *AB* into two parts in *D*, ſo that $AD : DB :: x : y$, and the weights of *AD* and *DB* will be reſpectively x and y;

y; biſect AD in E and DB in F; then ſince AD and DB keep the lever at reſt, they will keep it at reſt when they are collected at E and F (Art. 72); that is, x, when placed at E, will balance y, when placed at F; and $x : y :: AD : BD :: \frac{AD}{2} : \frac{BD}{2} :: \frac{AB-BD}{2} : \frac{AB-AD}{2} :: CB-BF : AC-AE :: CF : CE$.

Caſe 2. When the two forces act on the *ſame* ſide of the center of motion. Let A and B be two weights which balance each other upon the lever ACB, whoſe fulcrum is C, as in the 1ſt caſe; and ſuppoſe a power ſufficient to ſuſtain a weight equal to the ſum of the weights A and B, to be applied at C, in a direction oppoſite to that in which the weights act; then will this power ſupply the place of the fulcrum (Art. 70. Ax. 1); and A or B may be conſidered as the center of motion. Hence, when B is the center of motion, there is an equilibrium if the weight at A be to the power applied at $C :: A : A+B$; and from the 1ſt caſe, $A : B :: BC : AC$; therefore $A : A+B\ (C) :: BC : AB$.



74. Cor. 1. If two weights, or two forces, acting perpendicularly on the arms of a ſtraight lever, keep each other in equilibrio, they are inverſely as their diſtances from the center of motion.

For the weights will balance when they are in that proportion, and if the proportion be altered by increaſing or diminiſhing one of the weights, it's effort to turn the lever round will be altered, or the equilibrium will be deſtroyed.

75. Cor. 2. Since $A : B :: BC : AC$ when there is an equilibrium upon the lever AB, whoſe fulcrum is C, by multiplying extremes and means, $A \times AC = B \times BC$.

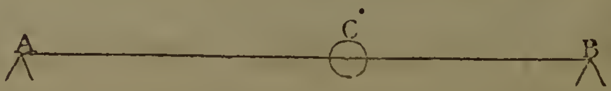
76. Cor. 3. When the power and weight act on the ſame ſide of the fulcrum, and keep each other in equilibrio, the weight ſuſtained by the fulcrum is equal to the difference between the power and the weight.

77. Cor. 4. In the common balance, the arms of the lever are equal; conſequently, the power and weight, or two weights, which ſuſtain each other, are equal. In the falſe balance, one arm is longer than the other; therefore the weight, which is ſuſpended at this arm, is proportionally leſs than the weight which it ſuſtains at the other.

78. Cor. 5. If the ſame body be weighed at the two ends of a falſe balance, it's true weight is a mean proportional between the apparent weights.

Call the true weight x, and the apparent weights, when it is ſuſpended at A and B, a and b reſpectively; then $a : x :: AC : BC$, and $x : b :: AC : BC$; therefore $a : x :: x : b$.

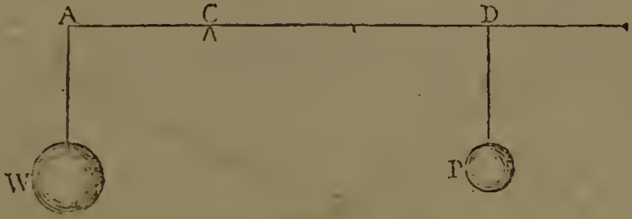
79. Cor. 6. If a weight C be placed upon a lever which is ſupported upon two props A and B in an



horizontal poſition, the preſſure upon A : the preſſure upon $B :: BC : AC$.

For if B be conceived to be the fulcrum, we have this proportion, the weight ſuſtained by A : the weight $C :: BC : AB$; in the ſame manner, if A be conſidered as the fulcrum, then the weight C : the weight ſuſtained by B :: AB : CA; therefore, ex æquo, the weight ſuſtained by A : the weight ſuſtained by B :: BC : AC.

80. Cor. 7. If a given weight P be moved along the graduated arm of a ſtraight lever, the weight W,



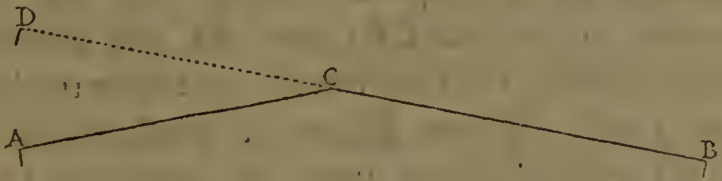
which it will balance at A, is proportional to CD, the diſtance at which the given weight acts.

When there is an equilibrium, $W \times AC = P \times DC$ (Art. 75); and AC and P are invariable; therefore $W \propto DC$ (Alg. Art. 199).

Prop. XV.

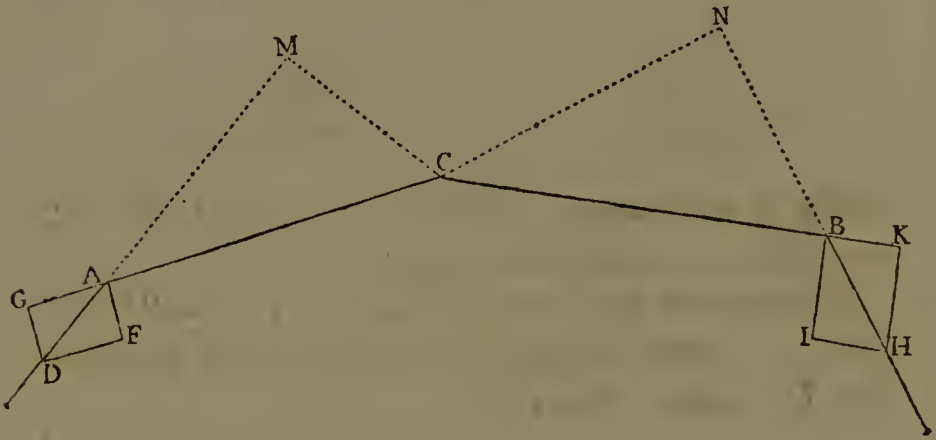
81. *If two forces, acting upon the arms of any lever, keep it at reſt, they are to each other inverſely as the perpendiculars drawn from the center of motion to the directions in which the forces act.*

Caſe 1. Let two forces, A and B, act *perpendicularly* upon the arms CA, CB, of the lever ACB whoſe fulcrum is C, and keep each other at reſt. Produce



BC to D and make $CD = CA$; then the effort of A to move the lever round C, will be the ſame, whether it be ſuppoſed to act perpendicularly at the extremity of the arm *CA*, or *CD* (Art. 70. Ax. 3); and on the latter ſuppoſition, ſince there is an equilibrium, *A* : *B* :: *CB* : *CD* (Art. 74); therefore *A* : *B* :: *CB* : *CA*.

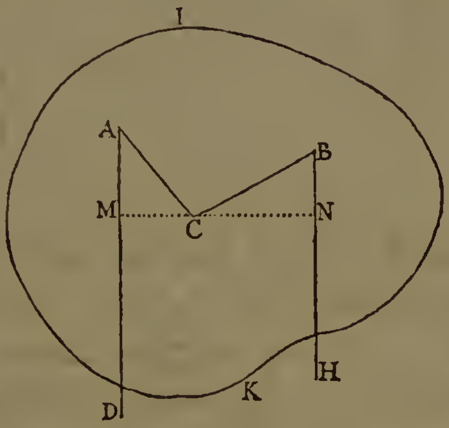
Caſe 2. When the directions *AD*, *BH*, in which the forces act, are not perpendicular to the arms. Take



AD and *BH*, to repreſent the forces; draw *CM* and *CN* at right angles to thoſe directions; alſo draw *AF* perpendicular, and *DF* parallel to *AC*, and complete the parallelogram *GF*; then the force *AD* is equivalent to the two *AF*, *AG*, of which, *AG* acts in the direction of the arm, and therefore can produce no effect in turning the lever round. Let *BH* be reſolved, in the ſame manner, into the two *BI*, *BK*, of which *BI* is perpendicular to, and *BK* in the direction of the arm *CB*; then *BK* will produce no effect in turning the lever round; and ſince the lever is kept at reſt, *AF* and *BI*, which produce this effect, and act perpendicularly upon the arms, are to each other, by the 1ſt caſe, inverſely as the arms; that is, *AF* : *BI* :: *CB* : *CA*, or $AF \times CA = BI \times CB$. Alſo, in the ſimilar triangles *ADF*, *ACM*, *AF* : *AD* :: *CM* : *CA*,

$: CA$, and $AF \times CA = AD \times CM$; in the ſame manner, $BI \times CB = BH \times CN$; therefore $AD \times CM = BH \times CN$, and $AD : BH :: CN : CM$.

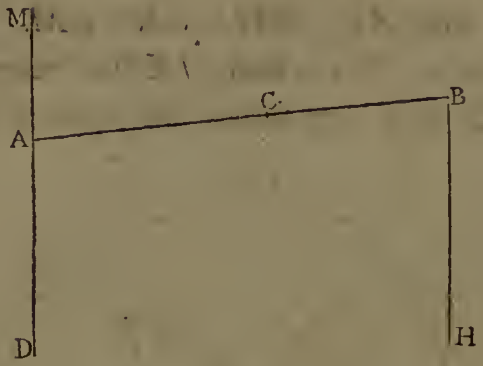
82. Cor. 1. Let a body IK be moveable about the center C, and two forces act upon it at A and B, in the directions AD, BH, which coincide with the plane ACB; join AC, CB; then this body may be conſidered as a lever ACB, and drawing the perpendiculars CM, CN, there will be an equilibrium, when the force acting at A : the force acting at B :: $CN : CM$.



83 Cor. 2. The effort of the force A, to turn the lever round, is the ſame, at whatever point in the direction MD it is applied; becauſe the perpendicular CM remains the ſame.

84. Cor. 3. Since $CA : CM ::$ rad. : ſin. CAM, $CM = \frac{CA \times \text{ſin.}\, CAM}{\text{rad.}}$; and, in the ſame manner, $CN = \frac{CB \times \text{ſin.}\, CBN}{\text{rad.}}$; ∴, when there is an equilibrium the power at A : the weight at B :: $\frac{CB \times \text{ſin.}\, CBN}{\text{rad.}} : \frac{CA \times \text{ſin.}\, CAM}{\text{rad.}} :: CB \times \text{ſin.}\, CBN : CA \times \text{ſin.}\, CAM$.

85. Cor. 4. If the lever *ACB* be ſtraight, and

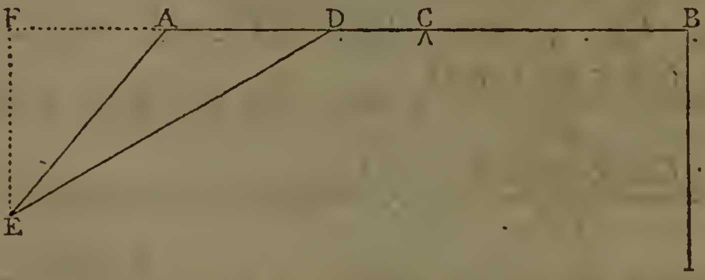


the directions *AD*, *BH*, parallel, $A : B :: BC : AC$; becauſe, in this caſe, ſin. $CAM =$ ſin. CBH.

86. Cor. 5. If two weights balance each other upon a ſtraight lever in any one poſition, they will balance each other in any other poſition of the lever; for the weights act in parallel directions, and the arms of the lever remain the ſame.

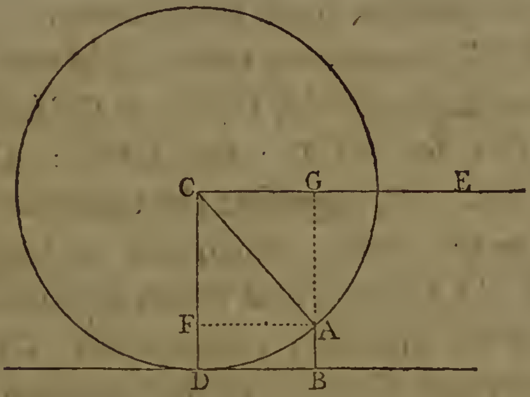
87. Cor. 6. If a man, balanced in a common pair of ſcales, preſs upwards by means of a rod, againſt any point in the beam, except that from which the ſcale is ſuſpended, he will preponderate.

Let the action upwards take place at *D*, then the ſcale, by the reaction downwards, will be brought



into the ſituation *E*; and the effect will be the ſame as if *DA*, *AE*, *DE* conſtituted one maſs; that is, drawing *EF* perpendicular to *CA* produced, as if the ſcale were applied at *F* (Art. 82); conſequently the weight, neceſſary to maintain the equilibrium, is greater, than if the ſcale were ſuffered to hang freely from *A*, in the proportion of *CF* : *CA*.

88. Cor. 7. Let *AD* repreſent a wheel, bearing a weight at it's center *C*; *AB* an obſtacle over which

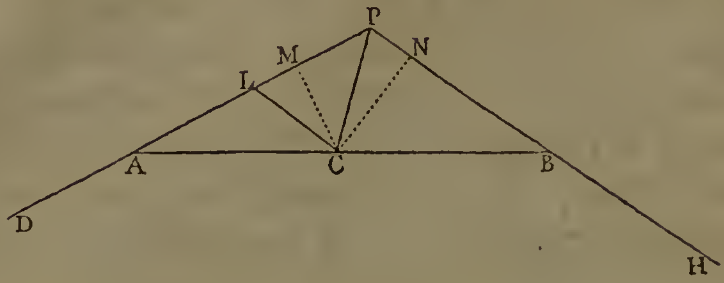


it is to be moved by a force acting in the direction *CE*; join *CA*, draw *CD* perpendicular to the horizon, and from *A* draw *AG*, *AF*, at right angles to *CE*, *CD*. Then *CA* may be conſidered as a lever whoſe center of motion is *A*, *CD* the direction in which the weight acts, and *CE* the direction in which the power is applied; and there is an equilibrium on this lever, when the power : the weight :: *AF* : *AG*.

Suppoſing the wheel, the weight, and the obſtacle given, the power is the leaſt when *AG* is the greateſt; that is, when *CE* is perpendicular to *CA*, or parallel to the tangent at *A*.

89. Cor. 8. Let two forces acting in the directions *AD*, *BH*, upon the arms of the lever *ACB*, keep each other in equilibrio; produce *DA* and *HB* till they meet in *P*; join *CP*, and draw *CL* parallel

to PB; then will PL, LC repreſent the two forces,

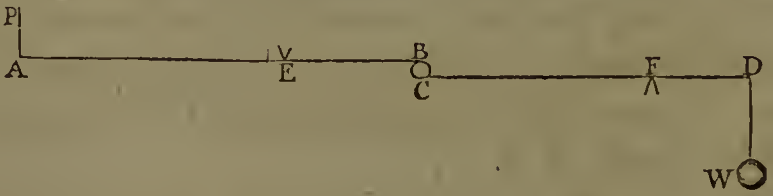


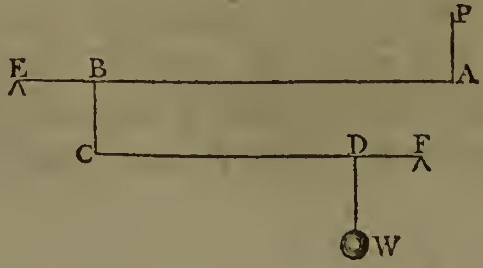
and PC the preſſure upon the fulcrum.

For, if PC be made the radius, CM and CN are the ſines of the angles CPM, CPN, or CPL, PCL; and $PL : LC :: \text{ſin.}\ PCL : \text{ſin.}\ LPC :: CN : CM$; therefore PL, LC, repreſent the quantities and directions of the two forces, which may be ſuppoſed to be applied at P (Art. 83), and which are ſuſtained by the reaction of the fulcrum; conſequently, CP repreſents the quantity and direction of that reaction (Art. 54), or PC repreſents the preſſure upon the fulcrum.

Prop. XVI.

90. *In a combination of ſtraight levers, AB, CD, whoſe centers of motion are E and F, if they act perpendicularly upon each other, and the directions in which the power and weight are applied be alſo perpendicular to the arms, there is an equilibrium when $P : W :: EB \times FD : EA \times FC$.*





For, the power at A : the weight at B, or C :: $EB : EA$; and the weight at C : the weight at D :: $FD : FC$, $\therefore P : W :: EB\times FD : EA\times FC$.

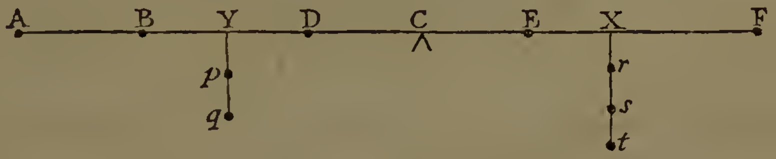
By the ſame method we may find the proportion between the power and the weight, when there is an equilibrium, in any other combination of levers.

91. Cor. If E and F be conſidered as the power and weight, A and D the centers of motion, we have, as before, $E : F :: FD\times BA : AE\times CD$. Hence the preſſure upon E : the preſſure upon F :: $FD\times BA : AE\times CD$.

Prop. XVII.

92. *Any weights will keep each other in equilibrio on the arms of a ſtraight lever, when the products, which ariſe from multiplying each weight by it's diſtance from the fulcrum, are equal, on each ſide of the fulcrum.*

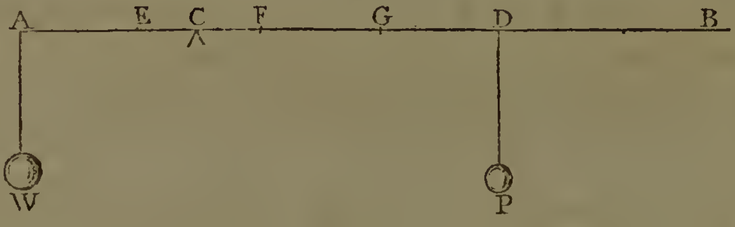
The weights A, B, D, and E, F, will balance each other upon the lever AF whoſe fulcrum is C, if $A\times AC+B\times BC+D\times DC=E\times EC+F\times FC$.



In CF take any point X, and let the weights r, s, t, placed at X, balance reſpectively, A, B, D; then $A\times AC=r\times XC$; $B\times BC=s\times XC$; $D\times DC=t\times XC$ (Art. 75); or, $A\times AC+B\times BC+D\times DC=\overline{r+s+t}\times XC$. In the ſame manner, let p and q, placed at Y, balance reſpectively, E and F; then $\overline{p+q}\times YC=E\times EC+F\times FC$; but by the ſuppoſition $A\times AC+B\times BC+D\times DC=E\times EC+F\times FC$; therefore $\overline{r+s+t}\times XC=\overline{p+q}\times YC$, and the weights r, s, t placed at X, balance the weights p, q, placed at Y (Art. 73); alſo A, B, D, balance the former weights, and E, F, the latter; conſequently A, B, D, will balance E and F.

93. Cor. 1. If the weights do not act in parallel directions, inſtead of the diſtances we muſt ſubſtitute the perpendiculars, drawn from the center of motion, upon the directions. (Vid. Art. 81).

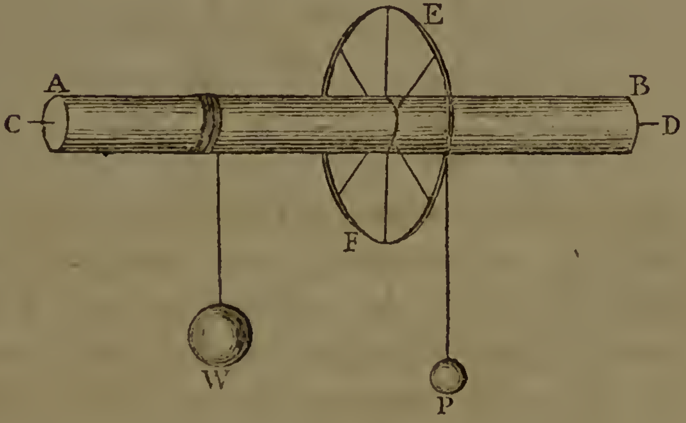
94. Cor. 2. In Art. 80 the lever is ſuppoſed to be without weight, or the arms AC, CD to balance each other: In the formation of the common *ſteel-yard* the longer arm CB is heavier than CA, and allowance muſt be made for this exceſs. Let the



moveable weight P, when placed at E keep the lever at reſt; then when W and P are ſuſpended upon the lever, and the whole remains at reſt, W ſuſtains P, and alſo a weight which would ſupport P when placed at E; therefore $W\times AC=P\times DC+P\times EC=P\times DE$; and ſince AC and P are invariable, $W\propto ED$; the graduation muſt therefore begin from E; and if P, when placed at *F*, ſupport a weight of one pound at *A*, take *FG*, *GD*, &c. equal to each other, and to *EF*, and when *P* is placed at *G* it will ſupport two pounds; when at *D* it will ſupport three pounds; &c.

ON THE WHEEL AND AXLE.

95. The *wheel and axle* conſiſts of two parts, a cylinder *AB* moveable about it's axis *CD*, and a circle *EF* ſo attached to the cylinder that the axis *CD* paſſes through it's center, and is perpendicular to it's plane.

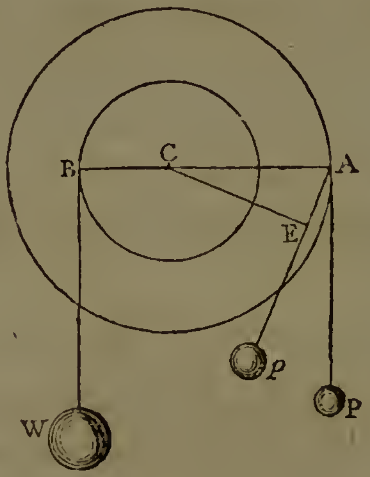


The power is applied at the circumference of the wheel, uſually in the direction of a tangent, and the weight is raiſed by a rope which winds round the axle.

Prop. XVIII.

96. *There is an equilibrium upon the wheel and axle, when the power is to the weight, as the radius of the axle to the radius of the wheel.*

The effort of the power to turn the machine round the axis, muſt be the ſame at whatever point in the axle the wheel is fixed; ſuppoſe it to be removed, and placed in ſuch a ſituation that the power and weight may act in the ſame plane, and let CA, CB, be the radii of the wheel and axle, at the extremities of which the power and weight act; then the machine becomes a lever ACB, whoſe center of motion is C, and ſince the radii CA, CB, are at right angles to AP and BW, we have $P : W :: CB : CA$ (Art. 82).



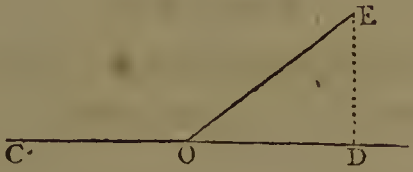
97. Cor. 1. If the power act in the direction Ap draw CE perpendicular to Ap, and there will be an equilibrium when $P : W :: CB : CE$ (Art. 82).

The ſame concluſion may alſo be obtained by reſolving the power into two, one perpendicular to AC, and the other parallel to it.

98. Cor. 2. If $2R$ be the thickneſs of the ropes by which the power and weight act, there will be an equilibrium when $P : W :: CB+R : CA+R$, ſince the power and weight muſt be ſuppoſed to be applied in the axes of the ropes.

The ratio of the power to the weight is greater in this caſe than the former; for if any quantity be added to the terms of a ratio of leſs inequality, that ratio is increaſed (Alg. Art. 162).

99. Cor. 3. If the plane of the wheel be inclined to the axle at the angle EOD, draw ED perpendicular to CD; and conſidering the wheel and axle as one maſs, there is an equilibrium when $P : W ::$ the radius of the axle : ED.



100. Cor. 4. In a combination of wheels and axles, where the circumference of the firſt axle is applied to the circumference of the ſecond wheel, by means of a ſtring, or by tooth and pinion, and the ſecond axle to the third wheel, &c. there is an equilibrium when $P : W ::$ the product of the radii of all the axles : the product of the radii of all the wheels. (Vid. Art. 90).

101. Cor. 5. When the power and weight act in parallel directions, and on *oppoſite* ſides of the axis, the preſſure upon the axis is equal to their *ſum*; and when they act on the *ſame* ſide, to their *difference*. In other caſes the preſſure may be eſtimated by Art. 89.

ON THE PULLEY.

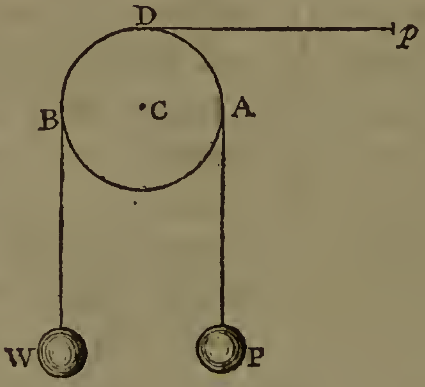
102. Def. A *Pulley* is a ſmall wheel moveable about it's center, in the circumference of which a groove is formed to admit a rope or flexible chain.

The pulley is ſaid to be *fixed*, or *moveable*, according as the center of motion is fixed or moveable.

Prop. XIX.

103. *In the ſingle fixed pulley, there is an equilibrium when the power and weight are equal.*

For whatever force is exerted at *D* in the direction *DAP*, by the power, an equal force is exerted by the weight in the direction *DBW*; theſe forces will therefore keep each other at reſt.

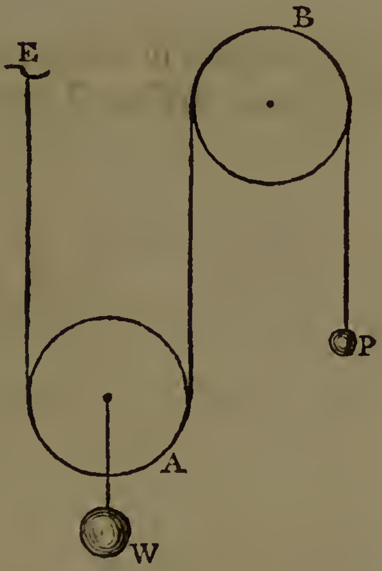


Cor. The propoſition is true when the power is applied in any other direction *Dp*; the only alteration made, by changing it's direction, is in the preſſure upon the center of motion. (Vid. Art. 106).

PROP. XX.

104. *In the ſingle moveable pulley, whoſe ſtrings are parallel, the power is to the weight as* 1 *to* 2.*

A ſtring fixed at *E*, paſſes under the moveable pulley *A*, and over the fixed pulley *B*; the weight is annexed to the center of the pulley *A*, and the power is applied at *P*. Then ſince the ſtrings *EA*, *BA* are in



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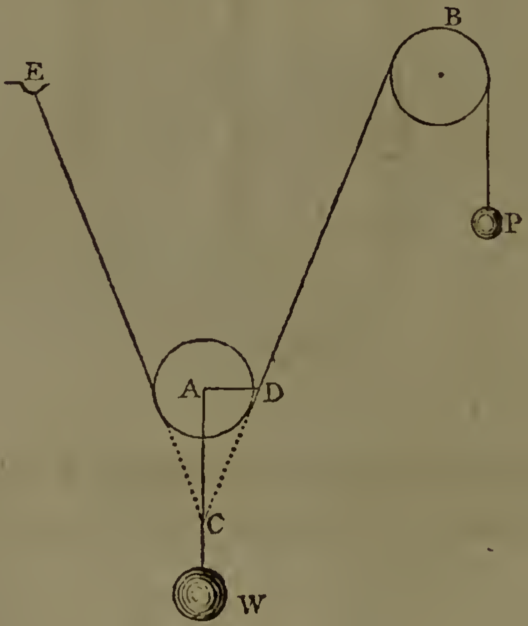
[*] In this and the following propoſitions, the power and weight are ſuppoſed to be in equilibrio.

in the direction in which the weight acts, they exactly ſuſtain it; and they are equally ſtretched in every point, therefore they ſuſtain it equally between them; or each ſuſtains half the weight. Alſo, whatever weight *AB* ſuſtains, *P* ſuſtains (Art. 103); therefore *P* : *W* :: 1 : 2.

Prop. XXI.

105. *In general, in the ſingle moveable pulley, the power is to the weight, as radius to twice the coſine of the angle which either ſtring makes with the direction in which the weight acts.*

Let *AW* be the direction in which the weight acts; produce *BD* till it meets *AW* in *C*, from *A* draw *AD*



at right angles to *AC*, meeting *BC* in *D*; then if *CD* be taken to repreſent the power at P, or the power which acts in the direction DB, CA will repreſent that part of it which is effective in ſuſtaining the weight, and AD will be counteracted by an equal and oppoſite force, ariſing from the tenſion of the ſtring CE; alſo, the two ſtrings are equally effective in ſuſtaining the weight; therefore $2AC$ will repreſent the whole weight ſuſtained; conſequently, $P : W :: CD : 2AC ::$ rad. : 2 coſ. DCA.

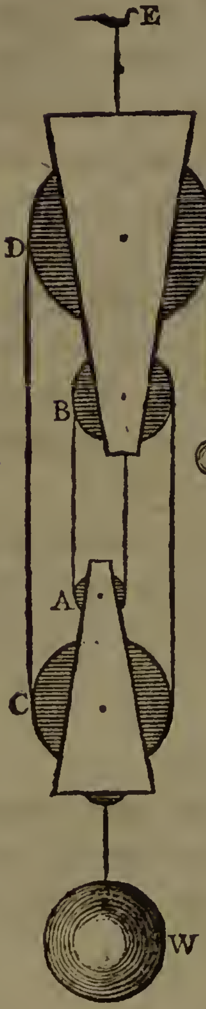
106. Cor. 1. If the figure be inverted, and E and B be conſidered as a power and weight which ſuſtain each other upon the fixed pulley A, W is the preſſure upon the center of motion; conſequently, the power : the preſſure :: radius : 2 coſ. DCA.

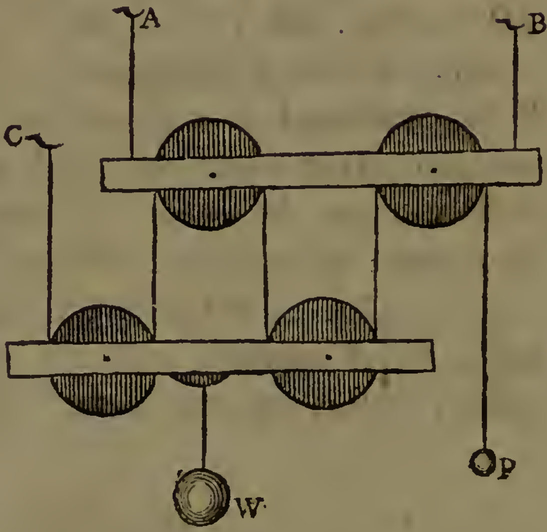
107. Cor. 2. When the ſtrings are parallel, the angle DCA vaniſhes, and it's coſine becomes the radius; in this caſe, the power : the preſſure :: 1 : 2.

Prop. XXII.

108. *In a ſyſtem where the ſame ſtring paſſes round any number of pulleys, and the parts of it between the pulleys are parallel,* $P : W :: 1$: *the number of ſtrings at the lower block.*

Since the parallel parts, or ſtrings at the lower block, are in the direction in which the weight acts, they exactly ſupport the whole weight; alſo, ſince there is an equilibrium, the tenſion in every point of theſe ſtrings is the ſame, and each of them ſuſtains an equal weight; conſequently, if there be n ſtrings, each ſuſtains $\frac{1}{n}$*th* part of the weight; hence, *P* ſuſtains $\frac{1}{n}$*th* part of the weight, or $P : W :: \frac{1}{n} : 1 :: 1 : n$.





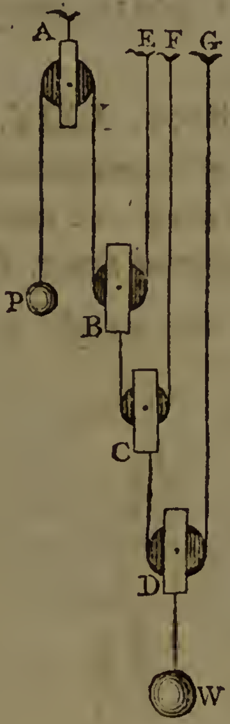
109. Cor. If two ſyſtems of this kind be combined, in which there are *m* and *n* ſtrings, reſpectively, at the lower blocks, $P : W :: 1 : mn$.

Prop.

Prop. XXIII.

110. *In a ſyſtem where each pulley hangs by a ſeparate ſtring, and the ſtrings are parallel,* $P : W :: 1 :$ *that power of* 2 *whoſe index is the number of moveable pulleys.*

In this ſyſtem, a ſtring paſſes over the fixed pulley *A*, and under the moveable pulley *B*, and is fixed at *E*; another ſtring is fixed at *B*, paſſes under the moveable pulley *C*, and is fixed at *F*; &c. in ſuch a manner that the ſtrings are parallel.



Then, by Art. 104, when there is an equilibrium,

P : the weight at B :: 1 : 2
the weight at B : the weight at C :: 1 : 2
the weight at C : the weight at D :: 1 : 2
&c.

Comp. $P : W :: 1 : 2 \times 2 \times 2 \times$ &c. continued to as many factors as there are moveable pulleys; that is, when there are n ſuch pulleys, $P : W :: 1 : 2^n$.

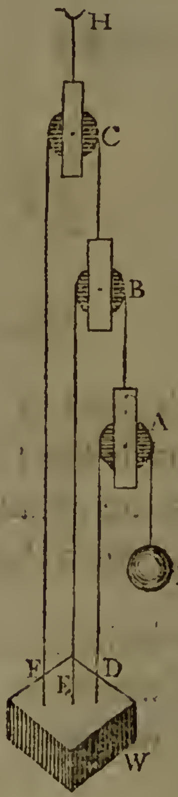
111. Cor. 1. The power and weight are wholly ſuſtained at A, E, F, G, &c. which points ſuſtain reſpectively, $2P$, P, $2P$, $4P$, &c.

112. Cor. 2. When the ſtrings are not parallel, $P : W ::$ rad. : 2 coſ. of the angle which the ſtring makes with the direction in which the weight acts, in each caſe (Art. 105).

Prop. XXIV.

113. *In a ſyſtem of n pulleys each hanging by a ſeparate ſtring, where the ſtrings are attached to the weight as is repreſented in the annexed figure,* $P : W :: 1 : 2^n - 1$.

A ſtring fixed to the weight at F paſſes over the pulley C, and is again fixed to the pulley B; another ſtring



ſtring fixed at E paſſes over the pulley B, and is fixed to the pulley A; &c. in ſuch a manner that the ſtrings are parallel:

Then, if P be the power, the weight ſuſtained by the ſtring DA is P; alſo the preſſure downwards upon A, or the weight which the ſtring BA ſuſtains, is $2P$ (Art. 107); therefore the ſtring EB ſuſtains $2P$; in the ſame manner, the ſtring FC ſuſtains $4P$; &c. and the whole weight ſuſtained is $P+2P+4P+$&c. Hence, $P : W :: 1 : 1+2+4+$&c. to n terms :: $1 : 2^n-1$ (Alg. Art. 222).

114. Cor. 1. Both the power and the weight are ſuſtained at H.

115. Cor. 2. When the ſtrings are not parallel, the power in each caſe, is to the correſponding preſſure upon the center of the pulley :: rad. : 2 coſ. of the angle made by the ſtring with the direction in which the weight acts (Art. 106). Alſo, by the reſolution of forces, the power in each caſe, or preſſure upon the former pulley, is to the weight it ſuſtains :: rad. : coſ. of the angle made by the ſtring with the direction in which the weight acts.

ON THE INCLINED PLANE.

Prop. XXV.

116. *If a body act upon a perfectly hard and ſmooth plane, the effect produced upon the plane is in a direction perpendicular to it's ſurface.*

Caſe 1. When the body acts perpendicularly upon the plane, it's force is wholly effective in that direction; ſince there is no cauſe to prevent the effect, or to alter it's direction.

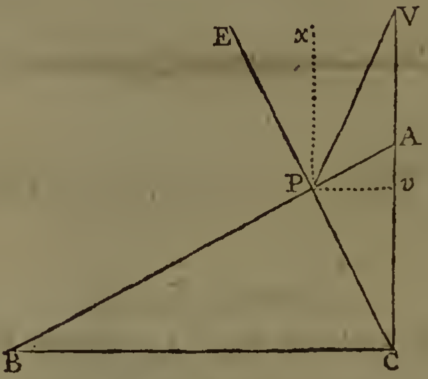
Caſe 2. When the direction in which the body acts is oblique to the plane, reſolve it's force into two, one parallel, and the other perpendicular, to the plane; the former of theſe can produce no effect upon the plane, becauſe there is nothing to oppoſe it in the direction in which it acts (Vid. Art. 29.); and the latter is wholly effective (by the firſt caſe); that is, the effect produced by the force is in a direction perpendicular to the plane.

117. Cor. The reaction of the plane is in a direction perpendicular to it's ſurface (Art. 32.).

PROP. XXVI.

118. *When a body is ſuſtained upon a plane which is inclined to the horizon,* $P : W ::$ *the ſine of the plane's inclination : the ſine of the angle which the direction of the power makes with a perpendicular to the plane.*

Let BC be parallel to the horizon, BA a plane inclined to it; P the body, ſuſtained at any point upon



the plane by a power acting in the direction PV. From P draw PC perpendicular to BA, meeting BC in C; and from C draw CV perpendicular to BC, meeting PV in V. Then the body P is kept at reſt by three forces which act upon it at the ſame time; the power, in the direction PV; gravity, in the direction VC; and the reaction of the plane, in the direction CP (Art. 117); theſe three forces are therefore properly repreſented by the three lines PV, VC and CP (Art. 55); or $P : W :: PV : VC ::$ ſin. PCV : ſin. VPC; and in the ſimilar triangles APC, ABC (Euc. 8. 6), the angles ACP, and CBA are equal; therefore $P : W ::$ ſin. ABC : ſin. VPC.

119. Cor. 1. When PV coincides with PA, or the power acts parallel to the plane, $P : W :: PA : AC :: AC : AB$.

120. Cor. 2. When PV coincides with Pv, or the power acts parallel to the baſe, $P : W :: Pv : vC :: AC : CB$; becauſe the triangles PvC, ABC are ſimilar.

121. Cor 3. When PV is parallel to CV the power ſuſtains the whole weight.

122. Cor. 4. Since $P : W ::$ ſin. ABC : ſin. VPC, by multiplying extremes and means, $P \times$ ſin. $VPC = W \times$ ſin. ABC; and if W, and the ſine of the $\angle ABC$ be invariable, $P \propto \frac{1}{\text{ſin. } VPC}$ (Alg. Art. 206); therefore, P is the leaſt when $\frac{1}{\text{ſin. } VPC}$ is the leaſt, or ſin. VPC the greateſt; that is, when ſin. VPC becomes the radius, or PV coincides with PA. Alſo, P is indefinitely great when ſin. VPC vaniſhes; that is, when the power acts perpendicularly to the plane.

123. Cor. 5. If P and the $\angle ABC$ be given, $W \propto$ ſin. VPC; therefore W will be the greateſt when ſin. VPC is the greateſt, that is, when PV coincides with PA. Alſo, W vaniſhes when the ſin. VPC vaniſhes, or PV coincides with PC.

124. Cor. 6. The power : the preſſure :: PV : PC :: ſin. PCV : ſin. PVC :: ſin. ABC : ſin. PVC.

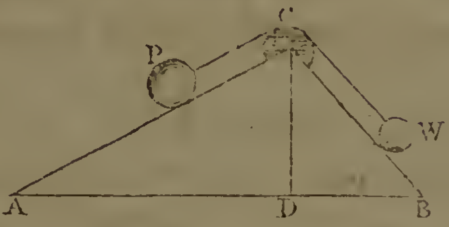
125. Cor. 7. When the power acts parallel to the plane, the power : the preſſure :: PA : PC :: AC : BC.

126. Cor. 8. When the power acts parallel to the baſe, the power : the preſſure :: Pv : PC :: AC : AB.

127. Cor. 9. $P \times$ ſin. $PVC =$ the preſſure $\times$ ſin. ABC; and when P and the $\angle ABC$ are given, the preſſure $\propto$ ſin. PVC; therefore, the preſſure will be the greateſt when PV is parallel to the baſe.

128. Cor. 10. When two ſides of a triangle, taken in order, repreſent the quantities and directions of two forces which are ſuſtained by a third, the remaining ſide, taken in the ſame order, will repreſent the quantity and direction of the third force (Art. 54). Hence, if we ſuppoſe PV to revolve round P, when it falls between Px which is parallel to VC, and PE, the direction of gravity remaining unaltered, the direction of the reaction muſt be changed, or the body muſt be ſuppoſed to be ſuſtained againſt the under ſurface of the plane. When it falls between PE and xP produced, the direction of the power muſt be changed: And when it falls between xP produced, and PC, the directions of both the power and reaction muſt be different from what they were ſuppoſed to be in the proof of the propoſition; that is, the body muſt be ſuſtained againſt the under ſurface of the plane, by a force which acts in the direction VP.

129. Cor. 11. If the weights P, W, ſuſtain each other upon the planes AC, CB, which have a common altitude CD, by means of a ſtring PCW which paſſes over the pulley C and is parallel to the planes, then $P : W :: AC : BC$.



For, ſince the tenſion of the ſtring is every where the ſame, the ſuſtaining power, in each caſe, is the ſame; and calling this power x,

$$P : x :: AC : CD \text{ (Art. 119)};$$
$$x : W :: CD : CB;$$
$$\text{comp. } P : W :: AC : CB.$$

ON THE WEDGE.

130. Def. A *Wedge* is a triangular priſm, or a ſolid generated by the motion of a plane triangle parallel to itſelf, upon a ſtraight line which paſſes through one of it's angular points. *

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* See alſo Euc. B. XI. Def. 13.

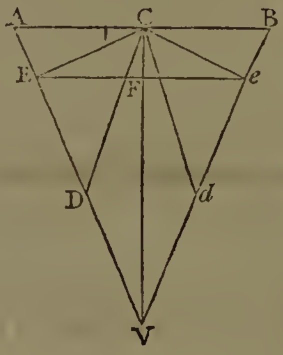
Knives, ſwords, coulters, nails, &c. are inſtruments of this kind.

The wedge is called *iſoſceles* or *ſcalene*, according as the generating triangle is *iſoſceles* or *ſcalene*.

PROP. XXVII.

131. *If two equal forces act upon the ſides of an iſoſceles wedge at equal angles of inclination, and a force act perpendicularly upon the back, they will keep the wedge at reſt, when the force upon the back is to the ſum of the forces upon the ſides, as the product of the ſine of half the vertical angle of the wedge and the ſine of the angle at which the directions of the forces are inclined to the ſides, to the ſquare of radius.*

Let AVB repreſent a ſection of the wedge, made by a plane perpendicular to it's ſides; draw VC perpendicular to AB; DC, dC, in the directions of the forces upon the ſides; and CE, Ce at right angles to AV, BV; join Ee, meeting CV in F.



Then, in the triangles VCA, VCB, ſince the angles VCA, CAV, are reſpectively equal to VCB, VBC, and VC is common to both, $AC = CB$, and the $\angle CVA = \angle CVB$. Again, in the triangles ACD, BCd, the angles DAC, CDA, are equal to the angles CBd, BdC, and $AC=BC$; therefore, $DC=dC$. In the ſame manner it may be ſhewn that $CE=Ce$, and $AE=Be$; hence the ſides AV, BV, of the triangle AVB, are cut proportionally in E and e; therefore Ee is parallel to AB (Euc. 2. 6), or perpendicular to CV; alſo, ſince $CE=Ce$ and CF is common to the right angled triangles CEF, CeF, we have $EF=eF$ (Euc. 47. 1).

Now ſince DC and dC are equal, and in the directions of the forces upon the ſides, they will repreſent them; reſolve DC into two, DE, EC, of which DE produces no effect upon the wedge, and EC, which is effective (Art. 116), does not wholly oppoſe the power, or force upon the back; reſolve EC therefore into two, EF, parallel to the back, and FC perpendicular to it, the latter of which is the only force which oppoſes the power. In the ſame manner it appears that eF, FC are the only effective parts of dC, of which FC oppoſes the power, and eF is counteracted by the equal and oppoſite force EF; hence, if $2CF$ repreſent the power, the wedge will be kept at reſt *; that is, when the force upon the back : the ſum of the reſiſtances upon the ſides :: $2CF : DC+dC :: 2CF : 2DC :: CF : DC$; and

$$CF : CE :: \text{ſin. } CEF : \text{rad.} :: \text{ſin. } CVE : \text{rad.}$$
$$CE : DC :: \text{ſin. } CDE : \text{rad.}$$
$$\text{Comp. } CF : DC :: \text{ſin. } CVE \times \text{ſin. } CDE : \overline{\text{rad.}}|^2.$$

132. Cor. 1. The forces do not ſuſtain *each other*, becauſe the parts DE, de are not counteracted.

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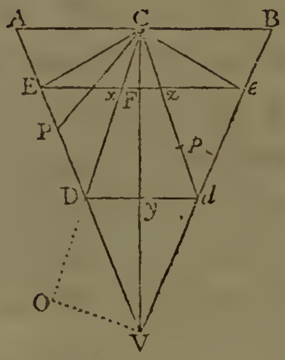
* The directions of the three forces muſt meet in a point, otherwiſe a rotatory motion will be given to the wedge.

133. Cor. 2. If the reſiſtances act perpendicularly upon the ſides of the wedge, the angle CDE becomes a right angle, and P : the ſum of the reſiſtances :: ſin. $CVE \times$ rad. : $\overline{\text{rad.}}^2$ :: ſin. CVE : rad. :: $AC : AV$.

134. Cor. 3. If the directions of the reſiſtances be perpendicular to the back, the angle $CDE = \angle CVE$, and P : the ſum of the reſiſtances :: $\overline{\text{ſin. } CVE}^2 : \overline{\text{rad.}}^2 :: AC^2 : AV^2$.

135. Cor. 4. When the reſiſtances act parallel to the back, ſin. $CDA =$ ſin. CAV, and P : the ſum of the reſiſtances :: ſin. $CVA \times$ ſin. CAV : $\overline{\text{rad.}}^2 :: CA \times CV : AV^2 :: CE \times AV$[*] $: AV^2 :: CE : AV$.

136. Cor. 5. In the demonſtration of the propoſition it has been ſuppoſed that the ſides of the wedge are perfectly ſmooth; if on account of the friction, or by any other means, the reſiſtances are *wholly* effective, join Dd, which will cut CV at right angles in y, and reſolve DC, dC into Dy, yC, dy, yC, of which Dy and dy deſtroy each other, and $2yC$ ſuſtains the power. Hence, the power : the ſum of the reſiſtances :: $2yC : 2DC :: yC : DC$ :: ſin. CDy or DCA : rad.



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[*] By ſimilar triangles, $CE : CA :: CV : AV$; therefore $CE \times AV = CA \times CV$.

137. Cor. 6. If Ee cut DC and dC in x and z, the forces, xC, zC, when wholly effective, and the forces DC, aC acting upon ſmooth ſurfaces, will ſuſtain the ſame power $2CF$.

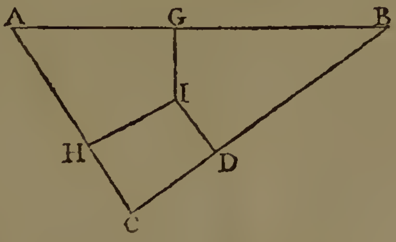
138. Cor. 7. If from any point P in the ſide AV, PC be drawn and the reſiſtance upon the ſide be repreſented by it, the effect upon the wedge will be the ſame as before; the only difference will be in the part PE which is ineffective.

139. Cor. 8. If DC be taken to repreſent the reſiſtance on one ſide, and pC, greater or leſs than dC, repreſent the reſiſtance on the other, the wedge cannot be kept at reſt by a power acting upon the back; becauſe, on this ſuppoſition, the forces which are parallel to the back are unequal.

This propoſition and it's corollaries have been deduced from the actual reſolution of the forces, for the purpoſe of ſhewing what parts are loſt, or deſtroyed by their oppoſition to each other; the ſame concluſions may, however, be very conciſely and eaſily obtained from Art. 142.

PROP. XXVIII.

140. *When three forces, acting perpendicularly upon the ſides of a ſcalene wedge, keep each other in equilibrio, they are proportional to thoſe ſides.*



Let GI, HI, DI, the directions of the forces, meet in *I*; then ſince the forces keep each other at reſt, they are proportional to the three ſides of a triangle which are reſpectively perpendicular to thoſe directions (Art. 59); that is, to the three ſides of the wedge.

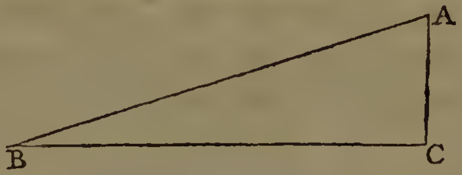
141. Cor. 1. If the lines of direction, paſſing through the points of impact, do not meet in a point, the wedge will have a rotatory motion communicated to it; and this motion will be round the center of gravity of the wedge. (Vid. Sect. V.)

142. Cor. 2. When the directions of the forces are not perpendicular to the ſides, the effective parts muſt be found, and there will be an equilibrium when thoſe parts are to each other as the ſides of the wedge.

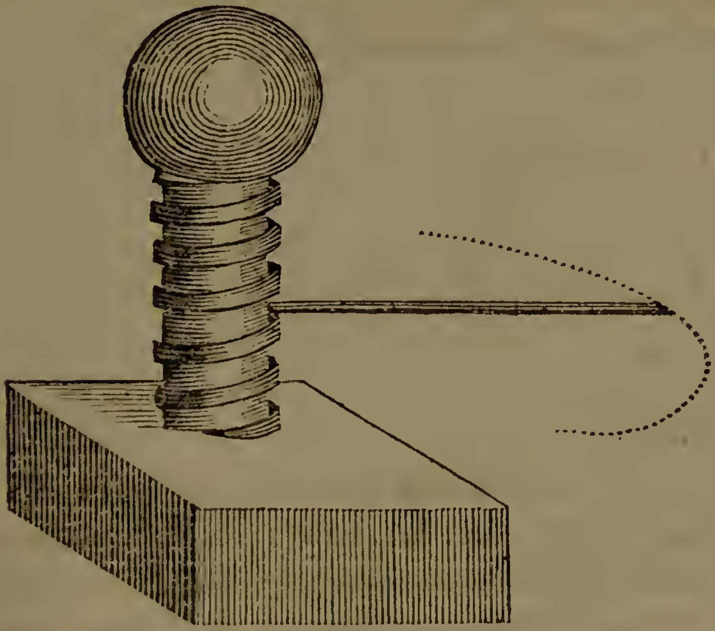
ON THE SCREW.

143. Def. The *Screw* is a mechanical power, which may be conceived to be generated in the following manner:

Let a ſolid and a hollow cylinder of equal diameters be taken, and let *ABC* be a plane triangle whoſe baſe *BC* is equal to the circumference of the ſolid cylinder; apply the triangle to this cylinder in ſuch a manner that the baſe *BC* may coincide with the baſe of the cylinder, and *BA* will form a ſpiral thread on it's ſurface. By applying to the cylinder, triangles, in ſucceſſion, ſimilar and equal to *ABC*, in ſuch a manner, that their baſes may be parallel to *BC*, the ſpiral thread may be continued; and ſuppoſing this thread to have thickneſs, or the cylinder to be protuberant where it falls, the external ſcrew will be formed, in which the



diſtance between two contiguous threads, meaſured in the direction of the axis of the cylinder, is *AC*. Again, let the triangles be applied in the ſame manner to the concave ſurface of the hollow cylinder, and where the thread falls let a groove be made, and the internal ſcrew will be formed. The two ſcrews being thus exactly adapted to each other, the ſolid or hollow cylinder, as the caſe requires, may be moved round

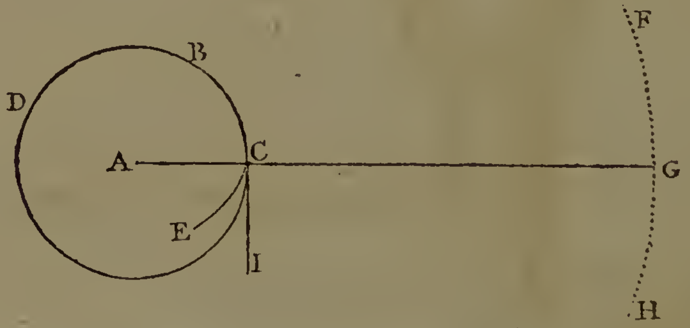


the common axis, by a lever perpendicular to that axis; and a motion will be produced in the direction of the axis, by means of the ſpiral thread.

Prop. XXIX.

144. *When there is an equilibrium upon the ſcrew, P : W :: the diſtance between two contiguous threads, meaſured in the direction of the axis : the circumference of the circle which the power deſcribes.*

Let *BCD* repreſent a ſection of the ſcrew made by a plane perpendicular to it's axis, *CE* a part of the ſpiral thread upon which the weight is ſuſtained; then *CE* is a portion of an inclined plane, whoſe height is the diſtance between two threads, and baſe equal to the circumference *BCD*. Call *F* the power which acting at *C* in the plane *BCD*, and in the direction *CI* perpendicular to *AC*, will ſuſtain the weight, or prevent the motion of the ſcrew round the axis; then ſince the weight is ſuſtained upon the inclined plane *CE* by a power *F* acting parallel to it's baſe; *F* : *W* :: the height : the baſe (Art. 120) :: the diſtance between two threads : the circumference *BCD*. Now, inſtead of ſuppoſing the power *F* to act at *C*, let a power *P* act perpendicularly at *G*, on the lever *GCA*, whoſe center of motion is *A*, and let this power produce the ſame effect at *C* that *F* does; then, by the property of the



the lever, $P : F :: CA : GA ::$ the circumference BCD : the circumference FGH. We have therefore theſe two proportions,

$F : W ::$ the diſtance between two threads : BCD
$P : F :: BCD : FGH$
comp. $P : W ::$ the diſtance between two threads : FGH.

145. Cor. 1. In the proof of this propoſition the whole weight is ſuppoſed to be ſuſtained at one point C of the ſpiral thread; if we ſuppoſe it to be diſperſed over the whole thread, then, by the propoſition, the power at G neceſſary to ſuſtain any part of the weight : that weight :: the diſtance between two threads : the circumference of the circle FGH; therefore the ſum of all theſe powers, or the whole power : the ſum of all the correſponding weights, or the whole weight, :: the diſtance between two threads : the circumference of the circle FGH (Alg. Art. 183).

146. Cor. 2. Since the power, neceſſary to ſuſtain a given weight, depends upon the diſtance between two threads and the circumference FGH, if theſe remain unaltered, the power is the ſame, whether the weight is ſuppoſed to be ſuſtained at C, or at a point upon the thread nearer to, or farther from, the axis of the cylinder.

147. Some Authors have deduced the properties of the mechanical powers *immediately* from the Third Law of Motion, contending that if the power and weight be ſuch as would ſuſtain each other, and the machine be put into motion, the momenta of the power and weight are equal; and conſequently, that the power × the velocity of the power = the weight

× the velocity of the weight; or *the power's velocity* : *the weight's velocity* :: *the weight* : *the power*.

Though this concluſion be juſt, the reaſoning by which it is attempted to be proved is inadmiſſible, becauſe the Third Law of Motion relates to the action of one body immediately upon another (Art. 36). It may however be deduced from the foregoing propoſitions; and as it is, in many caſes, the ſimpleſt method of eſtimating the power of a machine, it may not be improper to eſtabliſh it's truth.

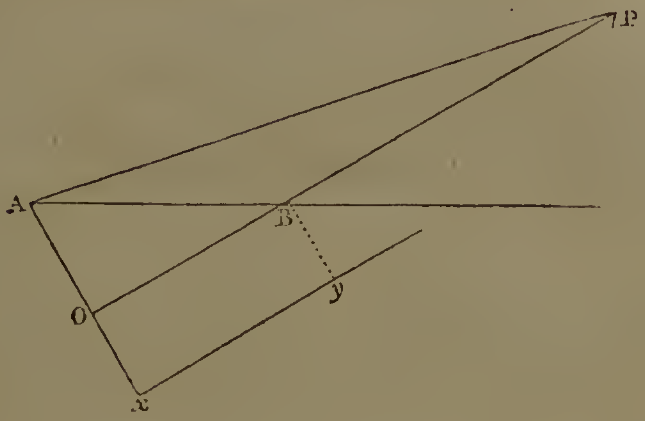
In the application of the rule, two things muſt be attended to: 1ſt. We muſt eſtimate the velocity of the power or weight in the direction in which it acts. 2dly. We muſt eſtimate that part only of the power or weight which is effective.

Theſe two conſiderations are ſuggeſted by the Second Law of Motion, according to which, motion is communicated in the *direction* of the force *impreſſed*, and is proportional to that force.

Prop. XXX.

148. *The velocity of a body in any one direction AB being given, to eſtimate it's velocity in any other direction BP.*

Suppoſe the motion of A to be produced by a force acting in the direction BP, by means of a ſtring which paſſes over a pulley at P; produce PB to O, making $PO = PA$; then OB is the ſpace which meaſures the approach of A to P. Now let the pulley be removed to ſuch a diſtance that the angle at P may be conſidered as evaneſcent, and the power will always act in the ſame direction BP; alſo, the angles at A and O are equal, and they are right angles, becauſe the three angles of the triangle APO are equal to two right angles, and the angle at P vaniſhes; therefore, the ſpace deſcribed in the direction OP, or BP, is determined by drawing AO perpendicular to OP. If the ſpace deſcribed in the direction xy, which is parallel to OP, be required, produce AO to x, and from B draw By at right angles to xy; then the figure $OByx$ is a parallelogram, and $OB = xy$ the ſpace required. Alſo, if the motion in the direction AB be uniform, the motion in the direction BP, or xy, is uniform; ſince $AB : OB ::$ rad. : coſ. ABO. Hence, the velocity in the direction AB : the velocity in the direction $BP :: AB : OB$ (Art. 11).

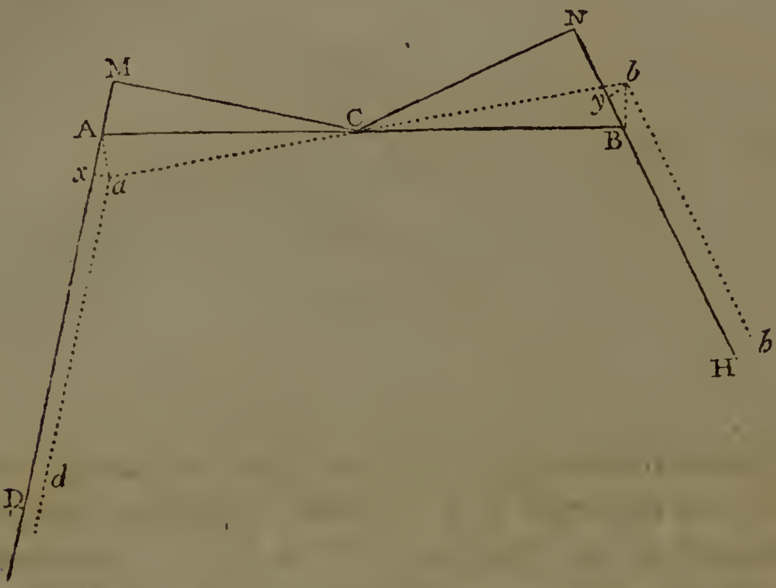


Prop. XXXI.

149. *If a power and weight ſuſtain each other on any machine, and the whole be put in motion, the velocity of the power : the velocity of the weight :: the weight : the power.*

Caſe

Caſe 1. *In the lever ACB*, let a power and weight, acting in the directions AD, BH, ſuſtain each other, and let the machine be moved uniformly round the center C, through a very ſmall angle ACa: Join Aa, Bb; draw CM, ax at right angles to MD; and CN, by at right angles to NB; then A's velocity : B's velocity :: Ax : By (Art. 148). Now the triangles Axa, MCA are ſimilar; becauſe $\angle xAC = \angle AMC + \angle MCA$ (Euc. 32. 1), and $\angle aAC = \angle AMC$; therefore, $\angle xAa = \angle MCA$; and the angles at M and x are right angles; conſequently, the remaining angles are equal; and



$$Ax : Aa :: CM : CA;$$

alſo, $Aa : Bb :: CA : CB$ from the ſim. △s ACa, BCb; and $Bb : By :: CB : CN$ from the ſim. △s Bby, BCN; com. $Ax : By :: CM : CN$ :: the weight : the power (Art. 81); or the power's velocity : the weight's velocity :: the weight : the power.

Caſe 2. *In the wheel and axle*, if the power be made to deſcribe a ſpace equal to the circumference of the wheel with an uniform motion, the weight will be uniformly raiſed through a ſpace equal to the circumference of the axle; hence, the power's velocity : the weight's velocity :: the circumference of the wheel : the circumference of the axle :: the radius of the wheel : the radius of the axle :: the weight : the power (Art. 96).

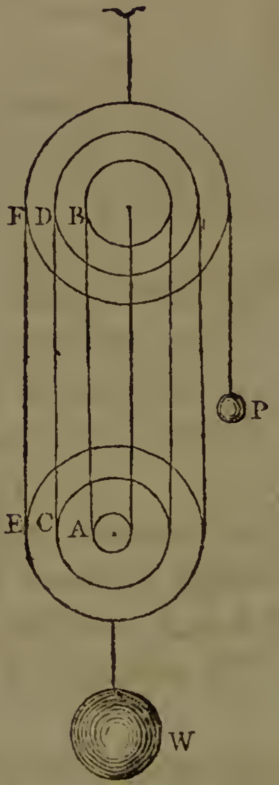
Caſe 3. *In the ſingle fixed pulley*, if the weight be uniformly raiſed 1 inch, the power will uniformly deſcribe 1 inch in the direction of it's action; therefore the power's velocity : the weight's velocity :: the weight : the power.

Caſe 4. *In the ſingle moveable pulley where the ſtrings are parallel*, if the weight be raiſed 1 inch, each of the ſtrings is ſhortened 1 inch, and the power deſcribes 2 inches; therefore, P's velocity : W's velocity :: W : P (Art. 104).

Caſe 5. *In the ſyſtem of pulleys deſcribed in Art.* 108, if the weight be raiſed 1 inch, each of the ſtrings at the lower block is ſhortened 1 inch, and the power deſcribes n inches; therefore, P's velocity : W's velocity :: W : P.

In this ſyſtem of pulleys, whilſt one inch of the ſtring paſſes over the pulley A, 2 inches paſs over the pulley B, 3 over C, 4 over D, &c.

Hence it follows, that if in the ſolid block A, the grooves A, C, E, &c. be cut, whoſe radii are 1, 3, 5, &c. and in the block B, the grooves B, D, F, &c. whoſe radii are 2, 4, 6, &c. and a ſtring be paſſed round theſe grooves as in the annexed figure; the grooves will anſwer the purpoſe of ſo many diſtinct pulleys, and every point in each, moving with the velocity of the ſtring in contact with it, the whole friction will be removed to the two centers of motion in the blocks *A* and *B*.



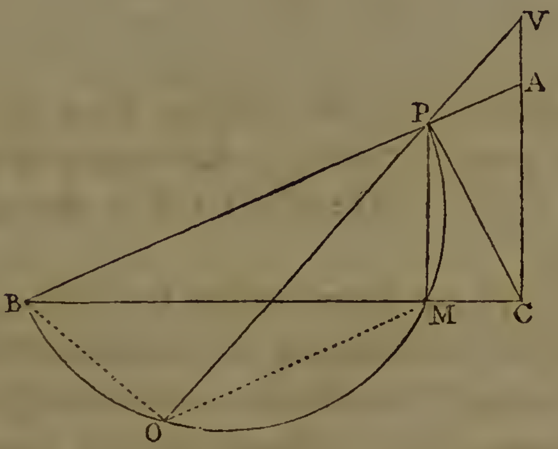
Caſe 6. *In the ſyſtem of pulleys deſcribed in Art.* 110, each ſucceeding pulley moves twice as faſt as the preceding;

therefore, *W*'s velocity : *C*'s velocity :: 1 : 2
C's velocity : *B*'s velocity :: 1 : 2
B's velocity : *P*'s velocity :: 1 : 2
&c.

comp. *W*'s velocity : *P*'s velocity :: 1 : 2 × 2 × 2 × &c. :: *P* : *W*.

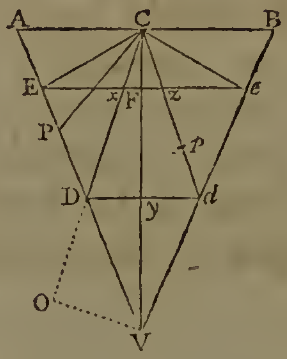
Caſe 7. *In the ſyſtem Art.* 113, if the weight be raiſed 1 inch the pulley *B* will deſcend 1 inch, and the pulley *A* will deſcend $2+1$ inches; in the ſame manner, the next pulley will deſcend $2\times\overline{2+1}+1$ inches, or $4+2+1$ inches; &c. therefore, *P*'s velocity : *W*'s velocity :: $1+2+4+$&c. : 1 :: *W* : *P*.

Caſe 8. Let a body be uniformly raiſed along *the inclined plane BA* from *B* to *P*, by a power acting parallel to *PV*; upon *BP* deſcribe a ſemicircle *BOP*,



cutting *BC* in *M*; produce *VP* to *O*, join *BO*, *PM*, *MO*. Then ſince the angles *BOP*, *BMP*, in the ſemicircle, are right angles, *OP* and *MP* are ſpaces uniformly deſcribed in the ſame time, by the power and weight in their reſpective directions (Art. 148); alſo, becauſe $\angle POM = \angle PBM = \angle PCV$, and $\angle OPM = \angle PVC$ (Euc. 29.1), the triangles *POM*, *PVC* are ſimilar, and *OP* : *MP* :: *VC* : *PV*, or the power's velocity : the weight's velocity :: the weight : the power, in the caſe of an equilibrium (Art. 118).

Caſe 9. *In the iſoſceles wedge*, *xC* is the only effective part of the reſiſtance *DÇ* (Vid. Art. 137); draw *VO* perpendicular to CD produced; then if the wedge be moved uniformly from C to V, CO is the ſpace uniformly deſcribed by the reſiſting force (Art. 148); hence, the power's velocity : the velocity of the reſiſting force :: $CV : CO :: Cx : CF$ :: the reſiſtance : the power.



Caſe 10. *In the ſcrew*, whilſt the power uniformly deſcribes the circumference of the circle FGH (Art. 144), the weight is uniformly raiſed through the diſtance between two contiguous threads; therefore P's velocity : W's velocity :: the circumference of the circle FGH : the diſtance between two threads :: $W : P$.

Caſe 11. *In any combination of the mechanical powers*, let $P : W$, $W : R$, $R : S$, &c. be the proportions between the power and weight in each caſe, when there is an equilibrium; then

P's velocity : W's velocity :: $W : P$
W's velocity : R's velocity :: $R : W$
R's velocity : S's velocity :: $S : R$
&c.
comp. P's velocity : S's velocity :: $S : P$.

SCHOLIUM.

SCHOLIUM.

150. It has been uſual to diſtinguiſh Levers into three kinds, according to the different ſituations of the power, weight, and center of motion; there are however only two kinds which eſſentially differ; thoſe in which the forces act on *contrary* ſides of the center of motion, as the common balance, ſteelyard, &c, and thoſe in which they act on the *ſame* ſide, as the ſtock knife, ſhears which act by a ſpring, oars, &c. the proportion between the forces, when there is an equilibrium, is expreſſed in the ſame terms in each caſe; but the levers differ in this reſpect, that the preſſure upon the fulcrum depends upon the ſum of the forces in the former caſe, and upon their difference in the latter; and conſequently, the friction upon the center of motion, cæteris paribus, is greater in the former caſe than the latter.

151. The Pulley has, by ſome Writers, been referred to the lever, and they have juſtly deduced it's properties from the proportions which are found to obtain in that mechanical power; for the adheſion of the pulley and the rope, which takes place at the circumference of the pulley, will overcome the friction at the center of motion; both becauſe it acts at a mechanical advantage, and becauſe the ſurface in contact is greater; and the friction depends, not only

upon the weight ſuſtained, but alſo upon the quantity of ſurface in contact: Thus, in practice, the rope and pulley move on together, and the pulley becomes a lever.

152. The Wedge has hitherto chiefly been applied to the purpoſes of ſeparating the parts of bodies, and it's power, notwithſtanding the friction, is much greater than the theory leads us to expect; the reaſon is, the effect is produced by impact, and the momentum thus generated is incomparably greater than the effect of preſſure, in the ſame time. Mr. ECKHARD, a very ingenious mechanic, by combining it with the wheel and axle, has conſtructed a machine, the power of which, conſidering it's ſimplicity, is much greater than that of any machine before invented.

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SECTION V.

ON THE CENTER OF GRAVITY.

153. Def. THE *Center of Gravity* of any body, or ſyſtem of bodies, is that point upon which the body or ſyſtem, acted upon only by the force of gravity, will balance itſelf in all poſitions *.

154. Hence it follows, that if a line or plane, which paſſes through the center of gravity, be ſupported, the body, or ſyſtem, will be ſupported.

155. Converſely, if a body, or ſyſtem, balance itſelf upon a line or plane, in all poſitions, the center of gravity is in that line or plane.

If not, let the line or plane be moved parallel to itſelf till it paſſes through the center of gravity, then we have increaſed both the quantity of matter on one ſide of the line or plane, and it's diſtance from the line or plane, and diminiſhed both, on the other ſide; hence, if the body balanced itſelf in all poſitions in the former caſe, it cannot, from the nature of the lever, balance itſelf in *all poſitions*, in the latter; conſequently, the center of gravity is not in this line, or plane (Art. 154), which is contrary to the ſuppoſition.

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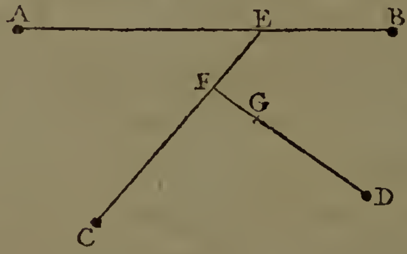
* That there is ſuch a point in every body, or ſyſtem of bodies, will be ſhewn hereafter.

156. Cor. By reaſoning in the ſame manner, it appears that a body, or ſyſtem of bodies, cannot have more than one center of gravity.

Prop. XXXII.

157. *To find the center of gravity of any number of particles of matter.*

Let A, B, C, D, &c. be the particles; and ſuppoſe A, B connected by the inflexible line AB without weight*; divide AB into two parts in E, ſo that



$A : B :: BE : EA$, or comp. $A+B : B :: AB : AE$; then will A and B balance each other upon E, or if E be ſupported, A and B will be ſupported in all poſitions (Art. 86); alſo the preſſure upon the point E is equal to the ſum of the weights A and B (Art. 70. Ax. 1). Join EC, and take $A+B : C :: CF : FE$, or $A+B+C : C :: EC : FE$; then if F be ſupported, E and C will be ſupported, that is, A, B and C will be ſupported, in all poſitions of the ſyſtem; and the preſſure

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* The particles muſt be ſuppoſed to be connected, otherwiſe they could not act upon each other, ſo as to balance round any point.

ſure upon F will be the ſum of the weights A, B and C. In the ſame manner, join FD, and divide it into two parts in G, ſo that $A+B+C : D :: DG : FG$, or $A+B+C+D : D :: FD : FG$, and the ſyſtem will balance itſelf in all poſitions upon G; that is, G is the center of gravity of the ſyſtem.

158. Cor. 1. From this propoſition it appears that every body, or ſyſtem of bodies, has a center of gravity.

159. Cor. 2. If the particles be ſuppoſed to be connected in any other manner, the ſame point G will be found to be their center of gravity (Art. 156).

160. Cor. 3. The effect of any number of particles in a ſyſtem, to produce or deſtroy an equilibrium, is the ſame, whether they are diſperſed, or collected in their common center of gravity.

161. Cor. 4. If A, B, C, &c. be bodies of finite magnitudes, G the center of gravity of the ſyſtem, may be found by ſuppoſing the bodies collected in their reſpective centers of gravity.

162. Cor. 5. If the bodies A, B, C, &c. be increaſed or diminiſhed in a given ratio, the ſame point G will be the center of gravity of the ſyſtem. For the points E, F, G, depend upon the relative, and not upon the abſolute weights of the bodies.

163. Cor. 6. If any forces which are proportional to the weights, act in parallel directions at A, B, C, D, they will ſuſtain each other upon the point G; and this point is ſtill called the center of gravity, though the particles are not acted upon by the force of gravity.

164. Cor. 7. A force applied at the center of gravity of a body cannot produce a rotatory motion in

it. For every particle reſiſts, by it's inertia, the communication of motion, and in a direction oppoſite to that in which the force applied tends to communicate the motion; theſe reſiſtances, therefore, of the particles act in parallel directions, and they are proportional to the weights (Art. 25); conſequently, they will balance each other upon the center of gravity.

Prop. XXXIII.

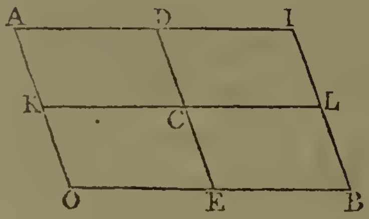
165. *To find the center of gravity of a right line* *.

The center of gravity of a right line compoſed of particles of matter which are equal to each other and uniformly diſperſed, is it's middle point. For, there are equal weights on each ſide, equally diſtant from the middle point, which will ſuſtain each other, in all poſitions, upon that point (Art. 86).

Prop. XXXIV.

166. *To find the center of gravity of a parallelogram.*

Let *AB* be an uniform lamina of matter in the form of a parallelogram; biſect *AO*, *AI* in *K* and *D*;



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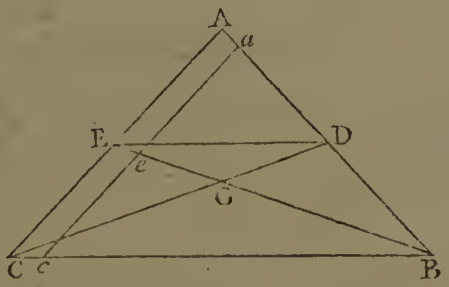
* When we ſpeak of a line or plane as having a center of gravity, we ſuppoſe it to be made up of equal particles of matter uniformly diffuſed over it.

D; draw *KL*, *DE* reſpectively parallel to *AI*, *AO*, cutting each other in *C*; this point *C* is the center of gravity of the figure. For if the parallelogram be ſuppoſed to be made up of lines parallel to *AI*, any one of theſe, as *KL*, is biſected by the line *DE* (ſince *AC*, *CI* are parallelograms, and therefore, $KC = AD = DI = CL$); conſequently, each line will balance itſelf upon *DE* (Art. 165), or the whole figure will balance itſelf upon *DE*, in all poſitions; therefore, the center of gravity is in that line (Art. 155). In the ſame manner it may be ſhewn that the center of gravity of the figure is in the line *KL*, conſequently *C*, the interſection of the two lines, is the center of gravity required.

Prop. XXXV.

167. *To find the center of gravity of a triangle.*

Let *ABC* be an uniform lamina of matter in the form of a triangle; biſect *AB*, *AC* in *D*, *E*; join *CD*, *BE*, cutting each other in *G*, this point is the center of gravity of the triangle.



Suppoſe the triangle to be made up of lines parallel to *CA*, of which let *cea* be one; then ſince the triangles *BEC*, *Bec* are ſimilar,

$BE : EC :: Be : ec$; alſo, in the Δ's BEA, Bea,
$AE : BE :: ea : Be$;

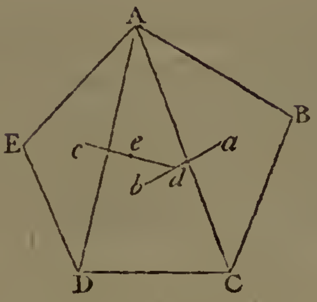
comp. $EA : EC :: ea : ec$; and $EA = EC$, therefore $ea = ec$; and conſequently, the line ac will balance itſelf in all poſitions upon BE. For the ſame reaſon every other line parallel to AC will balance itſelf, in all poſitions, upon BE, or the whole triangle will balance itſelf in all poſitions upon BE; therefore the center of gravity of the triangle is in that line. In the ſame manner it may be proved that the center of gravity is in the line CD; therefore it is in G, the interſection of the two lines BE, CD.

168. Cor. The diſtance of G from B is two thirds of the line BE. Join ED; then ſince $AD = DB$, and $AE = EC$, ED is parallel to BC (Euc. 2.6); therefore, the triangles AED, ACB are ſimilar, and $CB : CA :: ED : EA$, alt. $CB : ED :: CA : EA :: 2 : 1$. Alſo, the triangles CGB, EGD are ſimilar, therefore, $BG : CB :: EG : ED$, alt. $BG : GE :: CB : ED :: 2 : 1$; hence, $BG : BE :: 2 : 3$.

Prop. XXXVI.

169. *To find the center of gravity of any rectilinear figure.*

Let $ABCDE$ be the given figure. Divide it into the triangles ABC, ACD, ADE, whoſe centers of gravity a, b, c, may be found by the laſt propoſition; then if the triangles be collected in their reſpective centers of gravity (Art. 160), their common center of gravity may be found as in Prop. 32; that is, join ab and take $db : ad ::$ the triangle ABC : the triangle ADC,

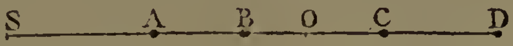


and d is the center of gravity of the two triangles ABC, ACD. Join dc, and take $ce : ed ::$ the ſum of the triangles ABC, ACD : the triangle AED, and e is the center of gravity of the figure.

Prop. XXXVII.

170. *To find the center of gravity of any number of bodies placed in a ſtraight line.*

Let A, B, C, D, be the bodies, collected in their reſpective centers of gravity; S any point in the ſtraight line SAD; O the center of gravity of all the bodies. Then ſince the bodies balance each other



upon O, $A\times AO+B\times BO=C\times CO+D\times DO$ (vid. Art. 92); that is, $A\times\overline{SO-SA}+B\times\overline{SO-SB}$ $=C\times\overline{SC-SO}+D\times\overline{SD-SO}$; hence, by mult. and tranſpoſition, $A\times SO+B\times SO+C\times SO+D\times$

$SO = A\times SA + B\times SB + C\times SC + D\times SD$; therefore,

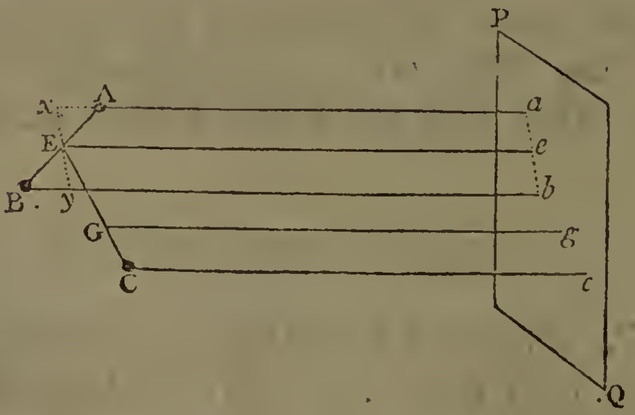
$$SO = \frac{A\times SA + B\times SB + C\times SC + D\times SD}{A + B + C + D}.$$

171. Cor. If any of the bodies lie the other way from S, their diſtances muſt be reckoned negative; and if SO be negative, the diſtance SO muſt be meaſured from S in that direction which, in the calculation, was ſuppoſed to be negative. (Vid. Alg. Art. 474).

Prop. XXXVIII.

172. *If perpendiculars be drawn from any number of bodies to a given plane, the ſum of the products of each body multiplied by it's perpendicular diſtance from the plane, is equal to the product of the ſum of all the bodies multiplied by the perpendicular diſtance of their common center of gravity from the plane.*

Let A, B, C, &c. be the bodies, collected in their reſpective centers of gravity; PQ the given



plane; draw Aa, Bb, Cc at right angles to PQ, and conſequently, parallel to each other (Euc. 6. 11); join AB, and take $AE : EB :: B : A$, then E is the center of gravity of A and B; through E draw Ee perpendicular to PQ, or parallel to Aa, and xEy perpendicular to Aa or Bb; then in the ſimilar triangles AEx, EBy, $Ax : AE :: By : BE$, alt. $Ax : By :: AE : BE :: B : A$; therefore $A \times Ax = B \times By$, that is, $A \times \overline{xa - Aa} = B \times \overline{Bb - yb}$, or ſince Ea, Eb are parallelograms, $A \times \overline{Ee - Aa} = B \times \overline{Bb - Ee}$; and by multiplication and tranſpoſition, $A \times Ee + B \times Ee = A \times Aa + B \times Bb$, that is, $\overline{A + B} \times Ee = A \times Aa + B \times Bb$.

Again, join EC, and take $CG : GE :: A + B : C$, then G is the center of gravity of the bodies A, B, C; draw Gg perpendicular to PQ; and it may be ſhewn, as before, that $\overline{A + B} \times Ee + C \times Cc = \overline{A + B + C} \times Gg$, or $A \times Aa + B \times Bb + C \times Cc = \overline{A + B + C} \times Gg$. The ſame demonſtration may be extended to any number of bodies.

173. Cor. 1. Hence $Gg = \dfrac{A \times Aa + B \times Bb + C \times Cc}{A + B + C}$; and if a plane be drawn parallel to PQ, and at the diſtance Gg from it, the center of gravity of the ſyſtem lies ſomewhere in this plane. In the ſame manner two other planes may be found, in each of which the center of gravity lies, and the point where the three planes cut each other is the center of gravity of the ſyſtem.

174. Cor. 2. If any of the bodies lie on the other ſide of the plane their diſtances muſt be reckoned negative.

175. Cor. 3. Wherever the bodies are ſituated,

if their reſpective perpendicular diſtances from the plane remain the ſame, the diſtance of their common center of gravity from the plane will remain the ſame.

176. Let the bodies lie in the ſame plane, and let perpendiculars Aa, Bb, Cc, Gg be drawn to any given *line* in that plane, then

$$Gg = \frac{A \times Aa + B \times Bb + C \times Cc}{A + B + C}.$$

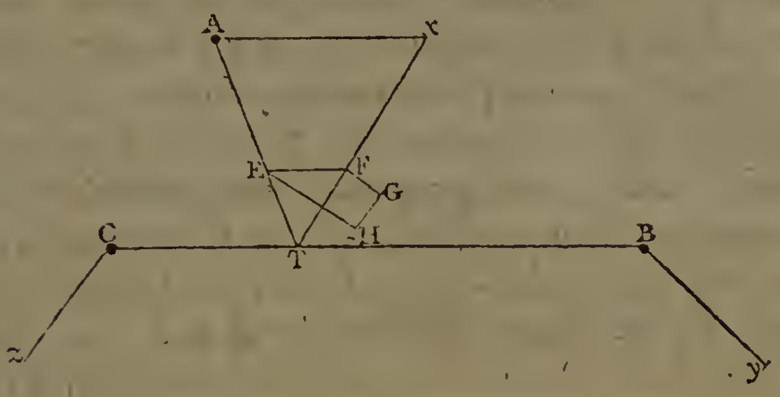
177. Cor. If A and B be on one ſide of the plane, and C on the other, and the plane paſs through the center of gravity, then $A \times Aa + B \times Bb = C \times Cc$. For $Gg \times \overline{A+B+C} = A \times Aa + B \times Bb - C \times Cc$, and $Gg = 0$, therefore $A \times Aa + B \times Bb - C \times Cc = 0$; or $A \times Aa + B \times Bb = C \times Cc$.

Prop. XXXIX.

178. *If any momenta be communicated to the parts of a ſyſtem, it's center of gravity will move in the ſame manner that a body, equal to the ſum of the bodies in the ſyſtem, would move, were it placed in that center, and the ſame momenta communicated to it, in the ſame directions.*

Let A, B, C be the bodies in the ſyſtem; join BC, and take $BT : TC :: C : B$; join AT, and take $TE : EA :: A : B + C$, or $TE : TA :: A : A + B + C$, then will E be the center of gravity of the ſyſtem (Art. 161).

Suppoſe the momentum communicated to A would cauſe it to move from A to x in T'', and at x let the body be ſtopped; join Tx, and take $TF : Tx :: A : A + B + C$, then F is the center of gravity of the bodies when they are at x, B, C; join EF, and ſince $TE : TA :: A : A+B+C :: TF : Tx$, EF is parallel to Ax (Euc. 2. 6), and conſequently the triangles TEF, TAx are



ſimilar; $\therefore EF : Ax :: A : A+B+C$. Hence if one body A in the ſyſtem be moved from A to x, the center of gravity is moved from E to F; which point may be thus determined; draw EF parallel to Ax, and take $EF : Ax :: A : A+B+C$. Next, let a momentum be communicated to B which would cauſe it to move from B to y in T''; at y let the body be ſtopped; then, according to the rule above laid down, draw FG parallel to By, and take $FG : By :: B : A+B+C$, and G will be the center of gravity of the bodies when they are at x, y, C. In the ſame manner let a momentum be communicated to C which would cauſe it to move from C to z in T'', and at z let the body be ſtopped; draw GH parallel to Cz and take $GH : Cz :: C : A+B+C$, then H is the center of gravity of the bodies when they are at x, y, z. If now the momenta, inſtead of being communicated ſeparately, be communicated *at the ſame inſtant* to the bodies, at the end of T'' they will be found in x, y, z reſpectively; therefore at the end of T'' their common center of gravity will be in H.

Now let E be a body equal to $A+B+C$, and let the ſame momentum be communicated to it that was before communicated to A, and in the ſame direction; then ſince EF is parallel to Ax, EF is in the *direction* in which the body E will move; alſo ſince the quantities of motion communicated to A and E are equal, their velocities are reciprocally proportional to their quantities of matter (Art. 19), or E's velocity : A's velocity :: $A : A+B+C :: EF : Ax$; therefore, EF and Ax are ſpaces deſcribed by E and A in equal times (Art. 11), or E will deſcribe the ſpace EF in T''. In the ſame manner FG is the ſpace which the body E will deſcribe in T'', if the momentum, before communicated to B, be communicated to it; and GH the ſpace it will deſcribe in T'', if the momentum before communicated to C, be communicated to it; join EH; and when the motions are communicated *at the ſame inſtant* to E, it will deſcribe EH in T'' (Art. 42). Hence it follows that when the ſame momenta are communicated to the parts of a ſyſtem, and to a body, equal to the ſum of the bodies, placed in the common center of gravity, this body and the center of gravity are in the ſame point at the end of T''; and T may repreſent any time; therefore, they are always in the ſame point.

The ſame demonſtration may be applied whatever be the number of bodies in the ſyſtem.

179. Cor. 1. If the parts of a ſyſtem move uniformly in right lines, the center of gravity will either remain at reſt, or move uniformly in a right line. For if the momenta communicated to the ſeveral parts of

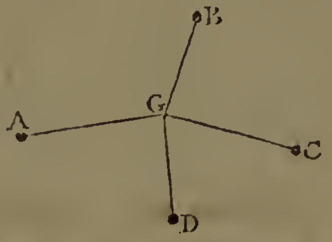
the ſyſtem be communicated to a body, equal to the ſum of the bodies, placed in the center of gravity of the ſyſtem, it will either remain at reſt or move uniformly in a right line (Art. 27).

180. If two weights ſupport each other upon any machine, and it be put in motion, the center of gravity of the weights will neither aſcend nor deſcend. For the momenta of the weights in a direction perpendicular to the horizon, are equal and oppoſite (Art. 149); therefore, if they were communicated to a body equal to the ſum of the bodies, placed in the common center of gravity, they would neither cauſe it to aſcend or deſcend.

181. Cor. 3. The motion or quieſcence of the center of gravity is not affected by the mutual actions of the parts of a ſyſtem upon each other. For action and reaction are equal and in oppoſite directions, and equal and oppoſite momenta communicated to a body, equal to the ſum of the bodies in the ſyſtem, will not diſturb it's motion or quieſcence.

182. Cor. 4. The effect of any force to communicate motion to the common center of gravity, is the ſame, upon whatever body in the ſyſtem it acts.

183. Cor. 5. If *G* be the center of gravity of the



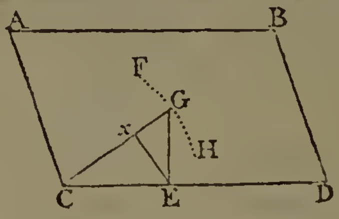
particles of matter *A*, *B*, *C*, *D*, which are acted upon only by their mutual attractions, they will meet at *G*. For they muſt meet, and their common center of gravity will remain at reſt (Art. 181); therefore, they muſt meet at that center.

184. Cor. 6. If a rotatory motion be communicated to a body and it be then left to move freely, the axis of rotation will paſs through the center of gravity. For the center of gravity itſelf, either remaining at reſt or moving uniformly forward in a right line, has no rotation.

Prop. XL.

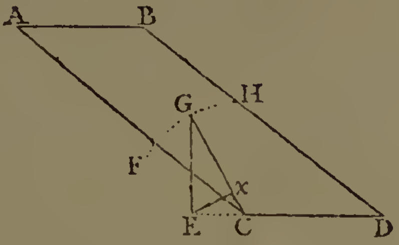
185. *If a body be placed upon an horizontal plane, and a line be drawn from it's center of gravity perpendicular to that plane, the body will be ſuſtained, or not, according as the perpendicular falls within or without it's baſe.*

Let *ABDC* repreſent the body, *G* it's center of gravity; draw *GE* perpendicular to the horizon; join *CG*, and with the radius *CG* deſcribe the circular arc *HGF*; then the body cannot fall over at *C* unleſs the



center of gravity deſcribes the circular arc *GF*. Suppoſe the whole force of gravity applied at *G* (Art. 160), and take *GE* to repreſent it; draw *Ex* perpendicular to CG; then the force GE is equivalent to the two Gx, xE, of which Gx cannot move the body either in the direction GF or GH; and when E falls within the baſe, xE acts at G in the direction GH; therefore the center of gravity cannot deſcribe the arc GF, that is, the body cannot fall over at C. In the ſame manner it may be ſhewn that it cannot fall over at D.

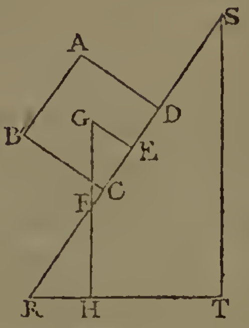
When the perpendicular GE falls without the baſe, xE acts in the direction GF, and ſince there is no



force to counteract this, the center of gravity will move in that direction, or the body will fall.

186. Cor. 1. In the ſame manner it may be ſhewn, that if a body be placed upon an inclined plane, and the lateral motion be prevented by friction, the body will be ſuſtained or not, according as the perpendicular to the horizon, drawn through it's center of gravity, falls within or without the baſe.

Ex. Let $ABDC$ repreſent a cube of uniform denſity placed upon the inclined plane RS; G it's center of gravity; draw GE perpendicular to CD, and GFH perpendicular to the horizon; then this body will not be ſuſtained upon the inclined plane, if the angle of the plane's inclination SRT, exceed half a right angle. For if the $\angle FRH$ be greater than half a right angle, the



$\angle RFH$ or GFE is leſs than half a right angle, and the $\angle FGE$ is greater than half a right angle; therefore, EF is greater than EG, or EC, and the body will roll.

187. Cor. 2. The higher the center of gravity of a body is, cæteris paribus, the more eaſily it is overturned.

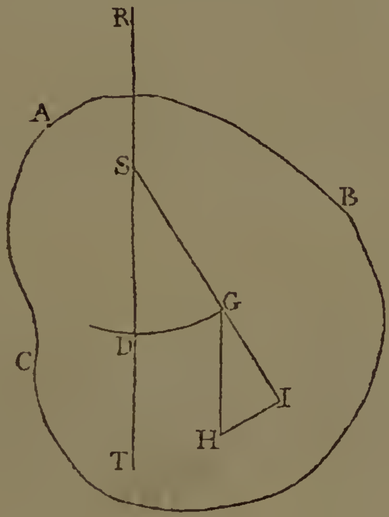
The ſame conſtruction being made as in the propoſition, the whole weight of the body : that part of the weight which keeps it ſteady upon it's baſe, or oppoſes any power employed to overturn it $:: GE : xE :: GC : CE$; and when CE and the whole weight of the body are given, the force which keeps the body ſteady $\propto \frac{1}{GC}$ (Alg. Art. 206); therefore as GC increaſes, that is, as GE increaſes, the force which keeps the body ſteady decreaſes, or the more eaſily will the body be overturned.

188. Cor. 3. When CE vaniſhes with reſpect to GC, the force which keeps the body ſteady vaniſhes, and the body may be overturned by a very ſmall force. Thus it is extremely difficult to balance a body upon a *point* placed *under* the center of gravity.

PROP. XLI.

189. *If a body be ſuſpended by any point, it will not remain at reſt till the center of gravity is in the line which is drawn through that point, perpendicular to the horizon.*

Let S be the point of ſuſpenſion of the body ABC;



G it's center of gravity; join SG and produce it; through S, and G, draw RST, and GH perpendiculars to the horizon; then the *immediate* effect of gravity is to draw the point G in the direction GH; take GH to repreſent the force in that direction, and draw HI perpendicular to GI; then the force GH is equivalent to the two GI, IH, of which GI is ſuſtained by the reaction of the point of ſuſpenſion S, and IH is employed in moving the center of gravity in a direction perpendicular to SG; therefore the center of gravity cannot remain at reſt till IH vaniſhes; that is, till the angle IGH, or GST vaniſhes, or SG coincides with RT.

190. Cor. Hence it follows that if a body be ſuſpended ſucceſſively by different points, and perpendiculars to the horizon be drawn through the points of ſuſpenſion, the center of gravity will lie in each of theſe perpendiculars, and conſequently, in the point of their interſection.

* * *

SECTION VI.

ON THE COLLISION OF BODIES.

191. Def. HARDNESS, which is found in different bodies in different degrees, conſiſts in a firm coheſion of the component particles; and that body is ſaid to be harder than another, whoſe particles require a greater force to ſeparate them. By a *perfectly hard* body we mean one whoſe parts cannot be ſeparated, or moved one amongſt another, by any finite force.

192. Def. The tendency in a body to recover it's former figure, after having been compreſſed, is called *elaſticity*. That body is ſaid to be more elaſtic than another, which recovers it's figure with the greater force, ſuppoſing the compreſſing force the ſame. By a *perfectly elaſtic* body we mean one which recovers it's figure with a force equal to that which was employed compreſſing it.

That ſuch a tendency exiſts in bodies is evident from a variety of experiments. If an ivory ball, ſtained with ink, be brought gently into contact with an unſtained ball, the ſpot received by the latter will be very ſmall, ſince two ſpheres touch each other only in a ſingle point; but if one of the balls be made to impinge upon the other, the ſpot will be enlarged; and the greater the force of impact, the greater will be the ſurface ſtained; hence it is manifeſt, that one, or both of the balls, has been compreſſed, and afterwards recovered it's ſpherical figure. Almoſt all bodies with which we are acquainted are elaſtic in a greater or leſs degree; but none perfectly ſo. In ſteel balls the force of elaſticity is to the compreſſing force as 5 to 9; in glaſs as 15 to 16; though in all caſes, the force of elaſticity ſeems to depend, in ſome meaſure, upon the diameter of the ball.

193. Def. The impact of two bodies is ſaid to be *direct*, when their centers of gravity move in the right line which paſſes through the point of impact.

In conſidering the effects of colliſion, the bodies are uſually ſuppoſed to be ſpheres of uniform matter; and in their actions upon each other, not to be affected by gravity, or any other force but that of inertia.

PROP. XLII.

194. *If the impact of two perfectly hard bodies be direct, after impact they will either remain at reſt, or move on uniformly, together.*

Since there is no force to turn either body out of the line of direction, they will continue in that line after impact *. Let A and B be the two bodies, moving in the *ſame* direction, and let A overtake B; then will A continue to accelerate B's motion, and B will continue to retard A's, till their velocities are equal, at which time they will ceaſe to act upon each other; and ſince there is no force to ſeparate them, they will move on together, and their common velocity, by the Firſt Law of Motion, will be uniform. When they move in *oppoſite* directions, if their forces be equal they will reſt after impact; if A's force be greater than B's, the whole velocity of B will be deſtroyed, and A's not being deſtroyed, A will communicate velocity to B, and B by it's reaction will retard A, till they move on together, as in the former caſe.

PROP. XLIII.

195. *If the impact of two perfectly hard bodies be direct, their common velocity may be found by dividing the whole momentum before impact, eſtimated in the direction of either motion, by the ſum of the quantities of matter.*

Let A and B be the quantities of matter contained in the bodies, a and b their velocities; then, when they move in the ſame direction, $Aa + Bb$ is the whole momentum in that direction, before impact. When they move in oppoſite directions, $Aa - Bb$ is the whole momentum eſtimated in the direction in which A moves.

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* The momenta of the particles in each body are proportional to their weights, ſince their velocities are equal; theſe momenta, therefore, will not turn the body to either ſide of the line paſſing through the center of gravity (Art. 163).

In the former caſe, as much as Aa, the momentum of A, is diminiſhed, ſo much is Bb, the momentum of B, increaſed by the impact (Art. 32); therefore $Aa + Bb$ is equal to the whole momentum after impact.

In the latter caſe, as much as Bb is diminiſhed by the impact, ſo much is Aa diminiſhed (Art. 32); and ſuppoſing Aa not to be leſs than Bb, the momentum remaining when B's momentum is deſtroyed, is $Aa - Bb$; and as much momentum as is afterwards communicated to B, ſo much is loſt by A; therefore $Aa - Bb$ is equal to the whole momentum after impact. If Aa be leſs than Bb, the momentum after impact, in the direction of B's motion, will be $Bb - Aa$; or, in the direction of A's motion, $Aa - Bb$.

Let v be the common velocity after impact; then $\overline{A+B} \times v$ is the whole momentum; conſequently, $\overline{A+B} \times v = Aa \pm Bb$, and $v = \frac{Aa \pm Bb}{A+B}$. In which expreſſion the poſitive ſign is to be uſed when the bodies move in the *ſame* direction before impact, and the negative ſign when they move in *oppoſite* directions.

196. Cor. 1. When the bodies move in oppoſite directions with equal momenta, they will remain at reſt after impact. In this caſe $Aa - Bb = 0$; $\therefore v = 0$.

197. Cor. 2. If Bb be greater than Aa, v is negative. This ſhews that the bodies will move in the direction of B's motion, which was ſuppoſed, in the propoſition, to be negative.

Prop. XLIV.

198. *In the direct impact of two perfectly hard bodies A and B, eſtimating the effects in the direction of A's motion,*

motion, $A+B : A ::$ *the relative velocity of the two bodies* : *the velocity gained by* B. *And* $A+B : B ::$ *their relative velocity* : *the velocity loſt by* A.

The ſame notation being retained; when the bodies move in the ſame direction, $a-b$ is their relative velocity (Art. 12); and v, their common velocity after impact, is $\frac{Aa+Bb}{A+B}$ (Art. 195); therefore, the velocity *gained* by B, or $v-b$, is $\frac{Aa+Bb}{A+B}-b$, or $\frac{Aa-Ab}{A+B}$; hence, $A+B : A :: a-b$: the velocity gained by B. Alſo, $a-\frac{Aa+Bb}{A+B}$, or $\frac{Ba-Bb}{A+B}$ is the velocity *loſt* by A; therefore $A+B : B :: a-b$: the velocity loſt by A.

When the bodies move in oppoſite directions, $a+b$ is their relative velocity (Art. 12); and $v=\frac{Aa-Bb}{A+B}$ (Art. 195); alſo, the velocity communicated to B upon the whole, in the direction of A's motion, is $b+v$, or $b+\frac{Aa-Bb}{A+B}$; that is, $\frac{Aa+Ab}{A+B}$; therefore, $A+B : A :: a+b$: the velocity gained by B.

The velocity loſt by A is $a-\frac{Aa-Bb}{A+B}$, or $\frac{Ba+Bb}{A+B}$; therefore, $A+B : B :: a+b$: the velocity loſt by A.

Ex. Let the weights of A and B be 10 and 6 *; their velocities 12 and 8, reſpectively; then, when they move in the ſame direction, $10+6 : 10 :: 12-8 : \frac{40}{16} = 2\frac{1}{2}$, the velocity gained by *B*; and $10+6 : 6 :: 12-8 : \frac{24}{16} = 1\frac{1}{2}$, the velocity loſt by *A*.

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* Vid. Art. 26.

When they move in oppoſite directions, $12+8$ is their relative velocity; and $10+6 : 10 :: 12+8 : \frac{200}{16} = 12\frac{1}{2}$, the velocity gained by *B* in the direction of *A*'s motion. Alſo, ſince it had a velocity 8 in the oppoſite direction before impact, it's velocity after impact is $4\frac{1}{2}$ in the direction of *A*'s motion. Again, $10+6 : 6 :: 12+8 : \frac{120}{16} = 7\frac{1}{2}$, the velocity loſt by *A*.

199. Cor. 1. Whilſt the relative velocity remains the ſame, the velocity gained by *B*, and the velocity loſt by *A*, are unaltered.

200. Cor. 2. Hence it alſo follows that the velocities, gained by *B*, and loſt by *A*, are the ſame whether both bodies are in motion, or *A* impinges upon *B* at reſt, with a velocity equal to their relative velocity in the former caſe.

201. Cor. 3. If the relative velocity be the ſame, the momentum communicated is the ſame, whether *A* impinges upon *B*, or *B* upon *A*.

Call r the relative velocity; then when *A* impinges upon *B*, $A+B : A :: r : \frac{Ar}{A+B}$, the velocity gained by *B*; therefore $\frac{ABr}{A+B}$ is the momentum gained by *B*. When *B* impinges upon *A*, $A+B : B :: r : \frac{Br}{A+B}$, the velocity gained by A; therefore $\frac{ABr}{A+B}$ is the momentum gained by A; which is alſo the momentum gained by B on the former ſuppoſition.

PROP. XLV.

202. *When the bodies are perfectly elaſtic, the velocity gained by the body ſtruck, and the velocity loſt by the ſtriking body, will be twice as great as if the bodies were perfectly hard.*

Let A and B be the bodies; then, as in Art. 194, A will accelerate B's motion, and B will retard A's, till their velocities are equal; and if they were perfectly hard they would then ceaſe to act upon each other, and move on together; thus, during the firſt part of the colliſion, the ſame effect is produced; that is, the ſame velocity is gained and loſt, as if the bodies were perfectly hard. But during this period the bodies are compreſſed by the ſtroke, and ſince they are, by the ſuppoſition, perfectly elaſtic, the force with which each will recover it's former ſhape is equal to that with which it was compreſſed; therefore, each body will receive another impulſe from the elaſticity equal to the former, or B will gain, and A loſe upon the whole, twice as great a velocity as if both bodies had been perfectly hard.

203. Cor. The ſame demonſtration may be applied to the caſe where one body is perfectly hard, and the other perfectly elaſtic.

PROP. XLVI.

204. *In the colliſion of two perfectly elaſtic bodies* A *and* B, $A+B : 2A ::$ *their relative velocity before impact : the velocity gained by* B *in the direction of* A*'s motion; and* $A+B : 2B ::$ *their relative velocity : the velocity loſt by* A, *in that direction.*

Call r the relative velocity of the bodies, x the velocity gained by B, and y the velocity loſt by A, when both bodies are perfectly hard; then $2x$ is the velocity gained by B, and $2y$ the velocity loſt by A, when they are perfectly elaſtic; and

$A+B : A :: r : x$ (Art. 198); therefore,
$A+B : 2A :: r : 2x$ (Alg. Art. 185), the velocity gained by B.

Again, $A+B : B :: r : y$ (Art. 198); therefore,
$A+B : 2B :: r : 2y$, the velocity loſt by A.

Ex. Let the weights of the bodies be 5 and 4, their velocities 7 and 5; then, when they move in the ſame direction, $5+4 : 10 :: 7-5 : \frac{20}{9} = 2\frac{2}{9}$, the velocity gained by B; therefore $5+2\frac{2}{9}$, or $7\frac{2}{9}$ is B's velocity after impact. Alſo, $5+4 : 8 :: 7-5 : \frac{16}{9} = 1\frac{7}{9}$, the velocity loſt by A; therefore $7-1\frac{7}{9}$, or $5\frac{2}{9}$, is A's velocity after impact. When they move in oppoſite directions, $5+4 : 10 :: 7+5 : \frac{120}{9} = 13\frac{1}{3}$, the velocity gained by B. Alſo, ſince it had a velocity 5 in the oppoſite direction, it's velocity after impact, in the direction of A's motion, is $13\frac{1}{3} - 5$, or $8\frac{1}{3}$. Again, $5+4 : 8 :: 7+5 : \frac{96}{9} = 10\frac{2}{3}$ A's velocity loſt; and ſince it had a velocity 7 before impact, after impact it will move in the oppoſite direction with a velocity $3\frac{2}{3}$.

205. Cor. 1. When $A=B$, the bodies interchange velocities. For, in this caſe, $A+B=2A=2B$; therefore, the velocity gained by B, and the velocity loſt by A, are reſpectively equal to their relative velocity before impact. Let a and b be their velocities before impact; then, when they move in the ſame direction, $a-b$ is the velocity gained by B, or loſt by A; therefore $a-b+b$, or a, is B's velocity after impact; and $a-\overline{a-b}$, or b, is A's velocity. If b be negative, or the bodies move in oppoſite directions, $a+b-b$, or a, is B's velocity, and $a-\overline{a+b}$, or $-b$, is A's velocity after impact.

206. Cor. 2. If the bodies move in oppoſite directions with equal quantities of motion, the whole momentum of each will be deſtroyed during the compreſſion, and an equal one generated by elaſticity in the oppoſite direction; each body will therefore be reflected with a velocity equal to that which it had before impact.

207. Cor. 3. In the congreſs of perfectly elaſtic bodies, the relative velocity after impact is equal to the relative velocity before impact.

Let a and b be the velocities of the bodies before impact; p and q their velocities after; then $a-b=q-p$. For, $A+B : 2A :: a-b : \frac{2A\times\overline{a-b}}{A+B}$, the velocity gained by B; $\therefore\ q=b+\frac{2A\times\overline{a-b}}{A+B}$. Alſo, $A+B : 2B :: a-b : \frac{2B\times\overline{a-b}}{A+B}$, the velocity loſt by A; therefore $p=a-\frac{2B\times\overline{a-b}}{A+B}$; and $q-p=b-a+\frac{\overline{2A+2B}\times\overline{a-b}}{A+B}$ $=b-a+2a-2b=a-b$. When the bodies A and B move in oppoſite directions, the ſign of b is negative; in other reſpects the demonſtration is the ſame.

208. Cor. 4. The ſum of the products of each body multiplied by the ſquare of it's velocity, is the ſame before and after impact.

The notation in the laſt article being retained; $Aa+Bb=Ap+Bq$ (Art. 34); by tranſpoſition, $Aa-Ap=Bq-Bb$; or $A\times\overline{a-p}=B\times\overline{q-b}$. Alſo $a-b=q-p$ (Art 207); or $a+p=q+b$; therefore $A\times\overline{a-p}\times\overline{a+p}=B\times\overline{q-b}\times\overline{q+b}$; or $Aa^2-Ap^2=Bq^2-Bb^2$; therefore $Aa^2+Bb^2=Ap^2+Bq^2$. If any of the quantities b, p, q, be negative, it's ſquare will be poſitive, and therefore the concluſion will not be altered.

209. Cor. 5. If there be a row of equal elaſtic bodies, A, B, C, D, &c. at reſt, and a motion be communicated to A, and thence to B, C, D, &c. they will all remain at reſt after the impact, except the laſt, which will move off with a velocity equal to that with which the firſt moved.

For A and B will interchange velocities (Art. 205); that is, A will remain at reſt, and B move on with A's velocity. In the ſame manner it may be ſhewn that all the others will remain at reſt after impact, except the laſt, which will move off with the velocity communicated to *A*.

210. Cor. 6. If the bodies decreaſe in magnitude they will all move in the direction of the firſt motion, and the velocity communicated to each ſucceeding body will be greater than that which was communicated to the preceding.

For, $A+B : 2B ::$ *A*'s velocity before impact : the velocity loſt by *A*; and ſince $2B$ is leſs than $A+B$, *A* does not loſe it's whole velocity; therefore it will move on after impact in the direction of the firſt motion. Alſo, $A+B : 2A ::$ *A*'s velocity before impact : the velocity gained by *B*; and ſince $2A$ is greater than $A+B$, the velocity gained by *B* is greater than *A*'s velocity before impact. In the ſame manner it may be ſhewn that *B*, *C*, *D*, &c. will move on in the direction of the firſt motion; and that the velocity communicated to each will be greater than that which was communicated to the preceding body.

211. Cor. 7. If the bodies increaſe in magnitude they will all be reflected back, except the laſt, and the velocity communicated to each ſucceeding body will be leſs than that which was communicated to the preceding.

For, in this caſe, $2B$ is greater than $A+B$; therefore, *A* loſes more than it's whole velocity, or it will move in the contrary direction. Alſo, $2A$ is leſs than $A+B$; therefore, the velocity gained by *B* is leſs than *A*'s velocity before impact. In the ſame manner it may be ſhewn that *B*, *C*, *D*, &c. will be reflected;

and that the velocity communicated to each will be leſs than that which was communicated to the preceding body.

212. Cor. 8. The velocity thus communicated from *A* through *B* to *C*, when *B* is greater than one of the two *A*, *C*, and leſs than the other, exceeds the velocity which would be communicated immediately from *A* to *C*.

Let a repreſent *A*'s velocity; then

$A+B : 2A :: \quad a \quad : \frac{2Aa}{A+B}$, the velocity of *B*;

and $B+C : 2B :: \frac{2Aa}{A+B} : \frac{2Aa}{A+B} \times \frac{2B}{B+C}$, the velocity communicated from *B* to *C*.

Again, $A+C : 2A :: a : \frac{2Aa}{A+C}$, the velocity communicated immediately from *A* to *C*. Hence it follows that the velocity communicated to *C*, by means of *B*, is greater than that which would be communicated to it immediately, if $\frac{2Aa}{A+B} \times \frac{2B}{B+C}$ be greater than $\frac{2Aa}{A+C}$; that is, if $A+C$ be greater than $\frac{\overline{A+B} \times \overline{B+C}}{2B}$, or $2A+2C$ greater than $A+C+B+\frac{AC}{B}$; or $A+C$ greater than $B+\frac{AC}{B}$. Suppoſe $A=B+x$, $C=B+y$; then $A+C=2B+x+y$, and $B+\frac{AC}{B}=B+\frac{B^2+Bx+By+xy}{B}=2B+x+y+\frac{xy}{B}$; therefore, the velocity communicated to *C* by means of *B*, is greater than the velocity communicated to it without B, if $2B+x+y$ be greater than $2B+x+y+\frac{xy}{B}$, which will always be the caſe when xy is negative, or when x and y have different ſigns; that is, when B is leſs than one of the bodies A, C, and greater than the other *.

213. Cor. 9. If the bodies be in geometrical progreſſion, the velocities communicated to them will be in geometrical progreſſion; and when there are n ſuch bodies, whoſe common ratio is r, the velocity of the firſt : the velocity of the laſt :: $\overline{1+r}\big|^{n-1} : 2^{n-1}$.

Let A, Ar, Ar^2, Ar^3, &c. be the bodies; a, b, c, d, &c. the velocities ſucceſſively communicated to them; then

$A+Ar : 2A :: a : b$, or
$1+r : 2 :: a : b$; and in the ſame manner,
$1+r : 2 :: b : c$
$1+r : 2 :: c : d$ &c.

therefore $a : b :: b : c :: c : d$ &c. Alſo, by compoſition, $\overline{1+r}\big|^{n-1} : 2^{n-1} :: a$: the velocity of the laſt.

214. Cor. 10. If the number of mean proportionals, interpoſed between two given bodies A and X, be increaſed without limit, the ratio of A's velocity to the velocity thus communicated to X will approximate to the ratio of $\sqrt{X} : \sqrt{A}$ as it's limit.

Let A, B, C, D, ... X be the bodies; a, b, c d, ... x the velocities communicated to them. Then

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* The velocity communicated from A through B to C, is a maximum when A, B, and C are in geometrical progreſſion (Flux. Art. 21. Ex. 9).

Then ſince the number of bodies interpoſed between A and X is increaſed without limit, their differences will be diminiſhed without limit; let $A+z=B$; then

$$2A+z : 2A :: a : b$$

$$\text{or } A+\frac{z}{2} : A :: a : b$$

$$^{*}\text{and } A+\frac{z}{2} : A :: \sqrt{A+z} : \sqrt{A} :: \sqrt{B} : \sqrt{A};$$

$$\text{therefore } \sqrt{B} : \sqrt{A} :: a : b$$

$$\text{in the ſame manner, } \sqrt{C} : \sqrt{B} :: b : c$$

$$\sqrt{D} : \sqrt{C} :: c : d$$

&c.

$$\text{comp. } \sqrt{X} : \sqrt{A} :: a : x.$$

PROP. XLVII.

215. *In the direct impact of two imperfectly elaſtic bodies A and B, if the compreſſing force be to the force of elaſticity* $:: 1 : m$, $A+B : \overline{1+m}\times A ::$ *their relative velocity before impact : the velocity gained by B in the direction of A's motion. And* $A+B : \overline{1+m}\times B ::$ *their relative velocity before impact : the velocity loſt by A, in that direction.*

By reaſoning as in Art. 202, it appears that the velocity gained by B, and the velocity loſt by A, during the compreſſion, are the ſame as if the bodies were perfectly hard; and the velocity communicated by the elaſticity is to the velocity communicated by the compreſſion $:: m : 1$. Call r the relative velocity before impact, x the velocity gained by B, and y the velocity loſt by A, during the compreſſion; then $\overline{1+m}\times x$ is the velocity gained by B, and $\overline{1+m}\times y$ the velocity loſt by A, upon the whole. Now

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* Since $\overline{A+\frac{z}{2}}|^{2} : A^{2} :: A^{2}+Az+\frac{z^{2}}{4} : A^{2} :: A+z+\frac{z^{2}}{4A} : A :: B+\frac{z^{2}}{4A} : A$, the ratio of $\overline{A+\frac{z}{2}}|^{2} : A^{2}$, when z is continually diminiſhed, approximates to the ratio of $B : A$, and conſequently, the ratio of $A+\frac{z}{2} : A$ approximates to the ratio of $\sqrt{B} : \sqrt{A}$ as it's limit.

$A+B : A :: r : x$ (Art. 198),

and $A+B : B :: r : y$;

therefore, $A+B : \overline{1+m}\times A :: r : \overline{1+m}\times x$, the velocity gained by B;

and $A+B : \overline{1+m}\times B :: r : \overline{1+m}\times y$, the velocity loſt by A.

216. Cor. 1. The relative velocity before impact : the relative velocity after impact $:: 1 : m$.

Let a and b be the velocities of the two bodies before impact, p and q their velocities after; then

$A+B : \overline{1+m}\times A :: a-b : \dfrac{\overline{1+m}\times A\times\overline{a-b}}{A+B}$, the velocity gained by B;

therefore, $q=b+\dfrac{\overline{1+m}\times A\times\overline{a-b}}{A+B}$;

in the ſame manner, $p=a-\dfrac{\overline{1+m}\times B\times\overline{a-b}}{A+B}$;

hence, $q-p=b-a+\dfrac{\overline{1+m}\times\overline{A+B}\times\overline{a-b}}{A+B}$, or $b-a+a-b+m\times\overline{a-b}$, i. e. $m\times\overline{a-b}$, is the relative velocity after impact; and $a-b : m\times\overline{a-b} :: 1 : m$.

When the bodies move in oppoſite directions the ſign of b is negative.

217. Cor. 2 Hence it appears that if the velocities of the bodies before and after impact be known, the elaſtic force is known.

218. Cor. 3. If A impinge upon B at reſt, A will remain at reſt after impact when $A : B :: m : 1$.

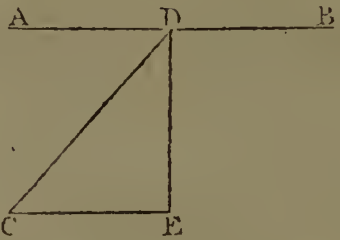
In this caſe A loſes it's whole velocity, and $A+B : \overline{1+m}\times B :: a :$ the velocity loſt by A; therefore $A+B=\overline{1+m}\times B$, and $A=mB$; conſequently, $A : B :: m : 1$.

219. Cor. 4. The momentum communicated is the ſame whether A impinges upon B, or B upon A, if the relative velocity be the ſame. This is the caſe when the bodies are perfectly hard (Art. 201); and the effect produced in elaſtic bodies is in a given ratio to that which is produced when the bodies are perfectly hard.

Prop. XLVIII.

220. *When a perfectly hard body impinges obliquely on a perfectly hard and immoveable plane AB, in the direction CD, after impact it will move along the plane, and the velocity before impact : the velocity after :: radius : the coſine of the angle CDA.*

Take CD to repreſent the motion of the body before



impact; draw CE parallel, and DE perpendicular to AB.

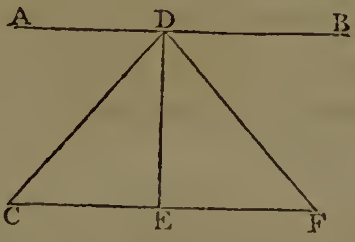
Then CD may be reſolved into the two CE, ED (Art. 43), of which ED is wholly employed in carrying the body in a direction perpendicular to the plane; and ſince the plane is immoveable, this motion will be wholly deſtroyed, (Vid. Art. 116). The other motion CE, which is employed in carrying the body parallel to the plane, will not be affected by the impact; and conſequently, there being no force to ſeparate the body and the plane, the body will move along the plane; and it will deſcribe $DB = CE$ in the ſame time that it deſcribed CD before impact; alſo, theſe ſpaces are uniformly deſcribed (Art. 27); conſequently, the velocity before impact : the velocity after :: $CD : CE$ :: radius : ſin. $\angle CDE$ :: radius : coſ. $\angle CDA$.

221. Cor. The velocity before impact : the difference between the velocity before and the velocity after, that is, the velocity loſt :: radius : rad. − coſ. $\angle CDA$ :: rad. : the verſed ſine of the angle CDA.

PROP. XLIX.

222. *If a perfectly elaſtic body impinge upon a perfectly hard, or perfectly elaſtic, immoveable plane* AB, *in the direction* CD, *it will be reflected from it in the direction* DF, *which makes, with* DB, *the angle* BDF *equal to the angle* ADC.

Let CD repreſent the motion of the impinging body; draw CF parallel, and DE perpendicular to AB;



AB; make $EF=CE$, and join DF. Then the whole motion may be reſolved into the two CE, ED, of which CE is employed in carrying the body parallel to the plane, and muſt therefore remain after the impact*; and ED carries the body in the direction ED, perpendicular to the plane; and ſince the plane is immoveable, this motion will be deſtroyed during the compreſſion, and an equal motion will be generated in the oppoſite direction by the force of elaſticity. Hence it appears that the body at the point D, has two motions, one of which would carry it uniformly from D to E, and the other from E to F, in the ſame time, viz. in the time in which it deſcribed CD before the impact; it will, therefore, deſcribe DF in that time (Art. 38). Alſo, in the triangles CDE, EDF, CE is equal to EF, the ſide ED is common, and the $\angle CED$ is equal to the $\angle DEF$; therefore, the $\angle CDE =$ the $\angle EDF$; hence, the $\angle CDA =$ the $\angle FDB$.

223. Cor. 1. Since $CD=DF$, and theſe are ſpaces uniformly deſcribed in equal times, before and after the impact, the velocity of the body after reflection is equal to it's velocity before incidence.

224. Cor. 2. If the body and plane be imperfectly elaſtic, take $DE : Dx ::$ the force of compreſſion : the force of elaſticity; draw fx parallel and equal to EF, join Ff, Df; then the two motions which the body has at D are repreſented by Dx, xf, and the body will deſcribe Df, after reflection, in the ſame time that it deſcribed CD

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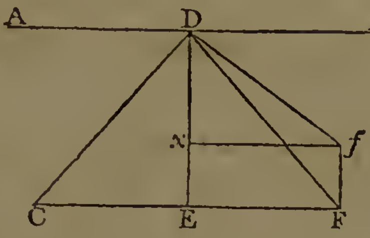
* Here we ſuppoſe the common ſurface of the body and plane, during the impact, to remain parallel to AB, in which caſe there is no cauſe to accelerate or retard the motion CE (Vid. Art. 116).

CD before incidence; therefore, the velocity before

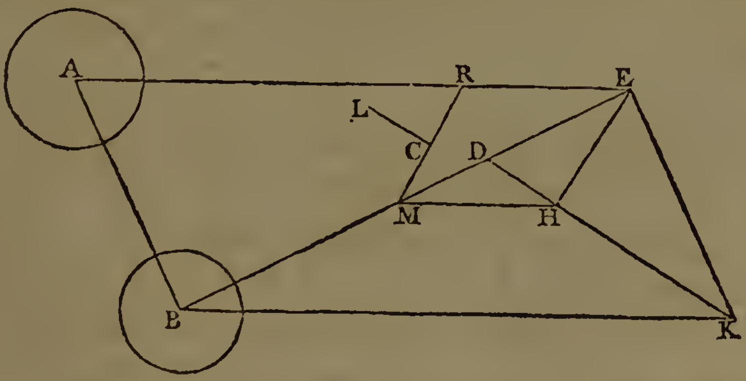


incidence : the velocity after reflection :: *CD* : *Df* :: *DF* : *Df* :: ſin. *DfF*, or ſin. of it's ſupplement the ∠ *EDf* : ſin. *DFf* or ſin. *FDE* :: ſin. *EDf* : ſin. *EDC*.

Prop. L.

225. *Having given the radii of two ſpherical bodies, their velocities, and the directions in which they move, to find the plane which touches them both at the point of impact.*

Let *AE*, *BE* be the directions in which the bodies move; and let *AE* and *BD* be ſpaces uniformly deſcribed by them in the ſame time; complete the parallelogram *ABKE*; join *KD*, and with the center *E* and radius equal to the ſum of the radii of the two bodies, deſcribe a circular arc cutting *KD* in *H*; join



EH, and complete the parallelogram *EHMR*: Then *R* and *M* will be the places of the centers of the two ſpheres when they meet; and if RC be taken equal to the radius of the ſphere A, the plane CL, which is drawn through C perpendicular to MR, will be the plane required.

Since MH is parallel to AE or BK, the triangles DMH, DBK are ſimilar, and $BK : BD :: MH : MD$; or $AE : BD :: RE : MD$; therefore $AE : BD :: AR : BM$ (Euc. 19. 5); and ſince AE and BD are ſpaces deſcribed in the ſame time by the uniform motions of A and B, AR and BM, which are proportional to them, will be deſcribed in the ſame time; when, therefore, the center of the body A is in R, the center of the body B is in M, and the diſtance $MR = HE =$ the ſum of the radii of the bodies; hence they will be in contact when they arrive at thoſe points. Alſo, MR which joins their centers will paſs through the point of contact; and LC will be a tangent to them both.

Prop. LI.

226. *Having given the motions, the quantities of matter, and the radii of two ſpherical bodies which impinge obliquely upon each other, to find their motions after impact.*

Let LN be the plane which touches the bodies at the point of impact; produce AB, which joins the centers of the bodies, indefinitely both ways; through the centers A and B, draw EAF, GBH parallel to LN; let CA, DB repreſent the velocities of the bodies before impact; reſolve CA into the two CI, IA*, of which CI is parallel, and IA perpendicular to LN; alſo reſolve DB

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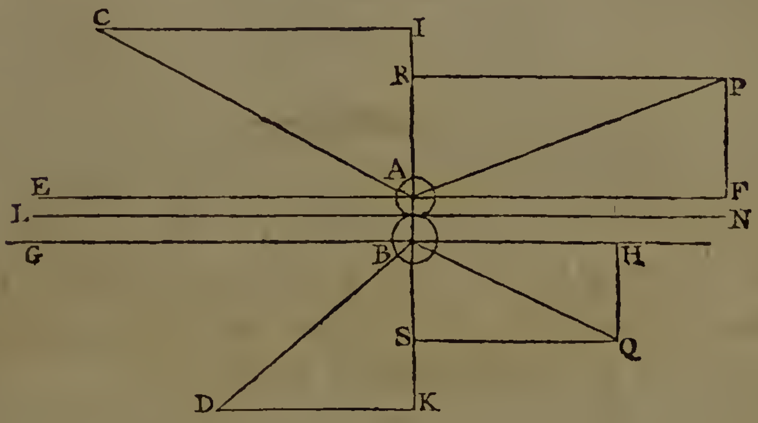
* Vid. Art. 43.

into two, *DK* parallel to *LN*, and *KB* perpendicular to it. Then *CA* and the angle *CAI*, which the direction of *A*'s motion makes with *AI* perpendicular to *LN*, being known, *CI* and *IA* are known; in the ſame manner *DK* and *KB* are known. Now *CI*, *DK*, which are parallel to the plane *LN*, will not be affected by the impact; and *IA*, *KB*, which are perpendicular to it, are the velocities with which the bodies impinge directly upon each other, and their effects may be calculated by Prop. 44, when the bodies are perfectly hard, and by Prop. 47, when they are elaſtic. Let *AR* and *BS* be the velocities of the bodies after impact, thus determined; take *AF*=*CI*, and *BH*=*DK*; complete the parallelograms *RF*, *SH* and draw the diagonals *AP*, *BQ*; then the bodies will deſcribe the lines *AP*, *BQ* after impact, and in the ſame time that they deſcribed *CA*, *DB* before impact.



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SECTION VII.

ON THE RECTILINEAR MOTIONS OF BODIES ACCELERATED OR RETARDED BY UNIFORM FORCES.

Prop. LII.

227. *IF a body be impelled in a right line by an uniform force, the velocity communicated to it is proportional to the time of it's motion* *.

The accelerating force is meaſured by the velocity uniformly generated in a given time (Art. 21), and in this caſe, the force is invariable, by the ſuppoſition; therefore, equal increments of velocity are always generated in equal times (Art. 20); if then in the time t the velocity a be generated, in the time mt the velocity ma is generated; that is, the velocity generated is proportional to the time (Alg. Art. 193).

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* By *force*, in this and the following propoſitions, we underſtand the *accelerating force*, no regard being paid to the quantity of matter moved, unleſs it be expreſsly mentioned. Alſo, the direction in which the force acts, to generate or deſtroy volocity, is ſuppoſed to coincide with the direction of the motion.

Prop. LIII

228. *If bodies be impelled in right lines by different uniform forces, the velocities generated in any times are proportional to the forces and times jointly.*

Let F and f be the forces, T and t the times of their action, V and v the velocities generated; alſo, let x be the velocity generated by the force f in the time T; then,

$V : x :: F : f$ (Art. 21);
$x : v :: T : t$ (Art. 227);
comp. $V : v :: FT : ft$; that is, the velocities generated are proportional to the forces and times jointly (Alg. Art. 195).

Ex. If a force repreſented by unity, generate a velocity repreſented by $2m$, in one ſecond of time, what velocity will the force F generate in T ſeconds?

Since $V \propto FT$, we have $1 \times 1 : FT :: 2m : 2mFT$, the velocity required.

229. Cor. Since $V \propto FT$, $T \propto \frac{V}{F}$ (Alg. Art. 205).

Prop. LIV.

230. *If a body's motion be retarded by an uniform force, the velocity deſtroyed in any time is equal to that which would be generated in the ſame time, were the motion accelerated by the ſame force.*

The

The force impreſſed is the ſame, by the ſuppoſition, whether the body move in the direction of the force, or in the oppoſite direction; therefore, the velocity generated in the former caſe, is equal to the velocity deſtroyed, in the ſame time, in the latter (Art. 29).

231. Cor. 1. Hence, the velocity deſtroyed by an uniform force is proportional to the time of it's action.

For, the velocity generated by the action of the force is in that ratio (Art. 227).

232. Cor. 2. The velocities deſtroyed by different uniform forces, are proportional to the forces and times jointly (Art. 228).

233. Cor. 3. If one body be projected in a direction oppoſite to that in which an uniform force acts, and another be moved at the ſame inſtant from a ſtate of reſt by the ſame force, the ſum of their velocities is always equal to the velocity with which the firſt was projected.

For, the velocity loſt by the former body, is equal to the velocity gained, in the ſame time, by the latter.

234. Cor. 4. If a body be projected in a direction oppoſite to that in which an uniform force acts, and at the ſame inſtant another be moved from a ſtate of reſt by the action of the ſame force, the ſum of the ſpaces deſcribed by the two bodies is equal to the ſpace which the former body would deſcribe in the ſame time, were it's firſt velocity continued uniform.

For, the ſum of the velocities of the two bodies, at any inſtant, is equal to the firſt velocity of the projected body; therefore, ſuppoſing them to move from the ſame point, the velocity with which they recede from each other, is equal to the velocity of the projected body; and conſequently, the ſum of the ſpaces deſcribed in any time by the two bodies, is equal to the ſpace which the firſt body would have deſcribed in the ſame time, had it moved uniformly forward with the velocity of projection.

235. Cor. 5. The ſpace deſcribed in any time by the projected body, is equal to the ſpace through which it would have moved with the firſt velocity continued uniform, diminiſhed by the ſpace through which it would have been moved from a ſtate of reſt, in the ſame time, by the action of the force.

Prop. LV.

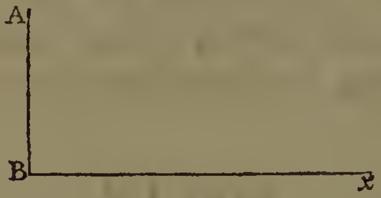
236. *If a body be moved through any ſpace, from a ſtate of reſt, by the action of an uniform force, and then be projected in the oppoſite direction with the velocity acquired, and move till that velocity is deſtroyed, the whole ſpaces deſcribed in the two caſes are equal.*

The velocity generated in any time, is equal to the velocity deſtroyed in the ſame time by the action of the ſame force (Art. 230), hence, the whole times of motion, in the two caſes, are equal; alſo, if equal times be taken, from the beginning of the motion in the former caſe, and from the end of the motion in the latter, the velocities at thoſe inſtants are equal. Since then the whole times of motion are equal, and alſo the velocities at all correſponding points of time, the whole ſpaces deſcribed are equal.

Prop. LVI.

237. *If a body be moved from a ſtate of reſt by the action of an uniform force, the ſpace deſcribed in any time is half that which would be deſcribed in the ſame time with the laſt velocity continued uniform.*

Let a body, acted upon by an uniform force, move from a ſtate of reſt at A and deſcribe AB in T''; alſo, let Bx be the ſpace which it would deſcribe in the ſame time with the laſt velocity continued uniform. At the ſame inſtant that this body begins to move from A, let another body be projected in the oppoſite direction with the laſt acquired velocity; then the ſum of the ſpaces deſcribed by the two bodies in T'' is equal to Bx (Art. 234); alſo, the ſum of the ſpaces deſcribed is $AB+BA$, or $2AB$ (Art. 236); therefore $2AB=Bx$, or $AB=\frac{Bx}{2}$ *.



Prop. LVII.

238. *When bodies are put in motion by uniform forces, the ſpaces deſcribed in any times, reckoning from the beginning of the motion in each caſe, are proportional to the times and laſt acquired velocities jointly.*

Let

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[*] The demonſtration of the propoſition on theſe principles was firſt given by Mr. Robertson, of Chriſt Church, Oxford. Vid. Hutton's Mathematical Dictionary, article *Acceleration.*

Let S and s be the ſpaces deſcribed in the times T and t; V and v the velocities acquired; then $2S$ and $2s$ are the ſpaces which would be deſcribed in the times T and t, with the uniform velocities V and v (Art. 237); and the ſpaces deſcribed with uniform velocities are proportional to the times and velocities jointly (Art. 14); hence,

$$2S : 2s :: TV : tv,$$

or $S : s :: TV : tv$ (Alg. Art. 184);

that is, $S \propto TV$ (Alg. Art. 195).

239. Cor. Hence, the times vary as the ſpaces directly and the laſt acquired velocities inverſely.

Prop. LVIII.

240. *The ſpaces deſcribed vary alſo as the forces and ſquares of the times; or as the ſquares of the velocities directly and the forces inverſely.*

In general, $S \propto TV$ (Art. 238): and $V \propto FT$ (Art. 228); hence, $TV \propto FT^2$ (Alg. Art. 203); $\therefore S \propto FT^2$. Alſo, $T \propto \frac{V}{F}$; therefore, $TV \propto \frac{V^2}{F}$, and conſequently, $S \propto \frac{V^2}{F}$.

241. Cor. 1. When F is given, $S \propto T^2 \propto V^2$. That is, when bodies are put in motion by the ſame, or equal uniform forces, the whole ſpaces deſcribed vary as the ſquares of the times, or as the ſquares of the laſt acquired velocities.

Ex. If a body be accelerated from a ſtate of reſt by an uniform force, and deſcribe m feet in the firſt ſecond of time, it will deſcribe $4m$, $9m$, $16m$, ... mT^2 feet, in the 2, 3, 4, ... T firſt ſeconds.

242. Cor. 2. If T be given, $S \propto F$; that is, the ſpaces deſcribed in equal times, by bodies which are put in motion by uniform forces, are proportional to thoſe forces.

243. Cor. 3. Since $S \propto \frac{V^2}{F}$, we have $V^2 \propto FS$ (Alg. Art. 203); that is, the ſquares of the velocities communicated are proportional to the forces and ſpaces deſcribed jointly.

244. Cor. 4. If V be given, $S \propto \frac{1}{F}$.

245. Cor. 5. When bodies in motion are retarded by uniform forces, and move till their whole velocities are deſtroyed, the ſpaces deſcribed vary as the forces and ſquares of the times; or, as the ſquares of the firſt velocities directly and the forces inverſely.

For, the time in which any velocity is deſtroyed, is equal to the time in which it would be generated by the ſame force; alſo, the ſpaces deſcribed, on ſuppoſition that the body in the latter caſe is moved from a ſtate of reſt, are equal (Art. 236); therefore, the ſame expreſſions which repreſent the relations of the forces, ſpaces, times, and velocities, in accelerated motions, repreſent them alſo when the motions are retarded, and the bodies move till their whole velocities are deſtroyed.

Thus, when bodies are made to impinge upon banks of earth, ſand, &c. where the retarding forces are invariable, the depths to which they ſink, or the whole ſpaces deſcribed, are as the ſquares of the firſt velocities directly and the forces inverſely; and the reſiſting forces are as the ſquares of the firſt velocities directly and the ſpaces inverſely.

PROP. LIX.

246. *If a body be moved from a ſtate of reſt by the action of an uniform force, the ſpaces deſcribed in equal ſucceſſive portions of time, reckoned from the beginning of the motion, are as the odd numbers* 1, 3, 5, 7, 9, *&c.*

If m be the ſpace deſcribed in the firſt portion of time, $4m$ will be the ſpace deſcribed in the two firſt portions (Art. 241); therefore $4m - m$, or $3m$, will be the ſpace deſcribed in the ſecond portion alone. Alſo $9m$ will be the ſpace deſcribed in the three firſt portions of time, and conſequently $9m - 4m$, or $5m$, will be the ſpace deſcribed in the third portion, &c. Thus the ſpaces deſcribed in the equal ſucceſſive portions of time are m, $3m$, $5m$, $7m$, $9m$, &c. which are as the odd numbers 1, 3, 5, 7, 9, &c.

247. Cor. When a body is retarded by an invariable force, the ſpaces deſcribed in equal portions of time, reckoning from the end of the motion, are as the odd numbers 1, 3, 5, 7, 9, &c.

For, when a body moves till it's whole motion is deſtroyed by an uniform force, the ſpace deſcribed in any time is equal to that which would be deſcribed in the correſponding time, were the body moved from a ſtate of reſt by the action of the ſame force (Vid. Art. 236).

PROP. LX.

248. *The force of gravity, at any given place, is an uniform force, which always acts in a direction perpendicular to the horizon, and accelerates all bodies equally.*

The ſame body will by it's gravity always produce the ſame effect under the ſame circumſtances; thus, it will, at the ſame place, bend the ſame ſpring; it will alſo fall through the ſame ſpace in the ſame time, if the reſiſtance of the air be removed; therefore, the force of gravity is uniform. Alſo, all bodies which fall freely by this force, deſcend in lines perpendicular to the horizon, and, in an exhauſted receiver, they *all* fall through the ſame ſpace in the ſame time; conſequently, gravity acts in a direction perpendicular to the horizon (Art. 29), and accelerates *all* bodies equally (Art. 242).

It is found by experiments made on the deſcent of heavy bodies, and on the oſcillations of bodies in ſmall circular arcs (Sect. VIII), that every body which falls freely in vacuo by the force of gravity, deſcends through $16\frac{1}{12}$ feet in one ſecond.

This fact being eſtabliſhed, every thing relating to the deſcent of bodies when they are accelerated by the force of gravity, and to their aſcent when they are retarded by that force, ſuppoſing the motions to be in vacuo, may be deduced from the foregoing propoſitions.

1ſt. When a body falls by the force of gravity, the velocity acquired in any time, as T'', is ſuch as would carry it uniformly over $2mT$ in $1''$; where $m = 16\frac{1}{12}$.

Since a body falls $16\frac{1}{12}$ feet in $1''$, it acquires a velocity which would carry it uniformly through $32\frac{1}{6}$ feet in $1''$ (Art. 237); and when a body is accelerated by a given invariable force, the velocity generated is proportional to the time (Art. 227); therefore $1'' : T'' :: 32\frac{1}{6} : 32\frac{1}{6}T$, the velocity acquired in T''; that is, the velocity acquired is ſuch as would carry the body uniformly over $32\frac{1}{6}\,T$ feet in $1''$. Let V be the velocity acquired, and $m = 16\frac{1}{12}$, then $V = 2mT$.

2d. The ſpace fallen through in T'', reckoned from the beginning of the motion, is mT^2 feet.

For $S \propto T^2$ (Art. 241); therefore, $1^2 : T^2 :: m : mT^2$, the ſpace deſcribed in T''. That is, $S = mT^2$.

Ex. 1. In $3''$ a body falls $9m$, or $9 \times 16\frac{1}{12} = 144\frac{3}{4}$ feet.

Ex. 2. In $\frac{1}{2}''$ a body falls $\frac{m}{4}$, or $16\frac{1}{12} \times \frac{1}{4} = 4\frac{1}{48}$ feet.

3d. The ſpace fallen through to acquire the velocity V, is $\frac{V^2}{4m}$ feet.

For, $S \propto V^2$ (Art. 241); therefore, $\overline{2m}^2 : V^2 :: m : S$, and $S = \frac{V^2}{4m}$ feet.

Ex. If a body fall from reſt till it acquires a velocity of 20 feet per ſecond, the ſpace fallen through is $\frac{20 \times 20}{64\frac{1}{3}}$, or 6.21 feet, nearly.

From the three preceding expreſſions, $V = 2mT$; $S = mT^2$; and $S = \frac{V^2}{4m}$; any one of the quantities S, T, V, being given, the other two may be found.

Ex. To find the time in which a body will fall 90 feet; and the velocity acquired.

Since $S=mT^2$, $T^2=\frac{S}{m}$, and $T=\sqrt{\frac{S}{m}}$; in this caſe, $T=\sqrt{\frac{90}{16\frac{1}{12}}}=2.36$ ſeconds, nearly.

Alſo, $S=\frac{V^2}{4m}$; therefore, $V=2\sqrt{mS}$; in this caſe, $V=2\sqrt{16\frac{1}{12}\times 90}=76$ feet per ſecond, nearly.

4th. If a body fall from reſt by the force of gravity, the ſpaces deſcribed in any equal ſucceſſive portions of time, reckoning from the beginning of the motion, are as the numbers 1, 3, 5, 7, &c. Thus, the ſpaces fallen through in the 1st, 2nd, 3rd, 4th ſeconds are $16\frac{1}{12}$, $3\times 16\frac{1}{12}$, $5\times 16\frac{1}{12}$, $7\times 16\frac{1}{12}$ feet, reſpectively. Alſo, if a body, projected upwards, move till it's whole velocity is deſtroyed, the ſpaces deſcribed in equal ſucceſſive portions of time are as the numbers 1, 3, 5, 7, &c. taken in an inverted order. Thus, if the velocity be wholly deſtroyed in 4″, the ſpaces deſcribed in the 1st, 2nd, 3rd, 4th ſeconds are $7\times 16\frac{1}{12}$, $5\times 16\frac{1}{12}$, $3\times 16\frac{1}{12}$, $16\frac{1}{12}$ feet, reſpectively.

5th. If a body begin to move in the direction of gravity with any velocity, the whole ſpace deſcribed in any time is equal to the ſpace through which the firſt velocity would carry the body, together with the ſpace through which it would fall by the force of gravity in that time.

For, the effect of *gravity* is the ſame with whatever velocity the body moves (Art. 29); that is, it's effect is the ſame whether the body falls from reſt or is projected with any velocity; and to this muſt be added the effect produced by the firſt velocity.

Ex. If a body be projected perpendicularly downwards with a velocity of 20 feet per ſecond, to find the ſpace deſcribed in 4″.

The ſpace deſcribed in 4″ by the firſt velocity is 4×20, or 80 feet; and the ſpace fallen through in 4″ by the action of gravity is $16\frac{1}{12} \times 16$, or $257\frac{1}{3}$ feet; therefore, the whole ſpace deſcribed is $80 + 257\frac{1}{3}$, or $337\frac{1}{3}$ feet.

6th. If a body be projected perpendicularly upwards, the height to which it will aſcend in any time is equal to the ſpace through which it would move with the firſt velocity continued uniform, diminiſhed by the ſpace through which it would fall by the action of gravity in that time (Art. 235).

Ex. 1. To what height will a body riſe in 3″, which is projected perpendicularly upwards with a velocity of 100 feet per ſecond?

The ſpace which the body would deſcribe in 3″, with the firſt velocity, is 300 feet; and the ſpace through which the body would fall by the force of gravity in 3″, is $16\frac{1}{12} \times 9$, or $144\frac{3}{4}$ feet; therefore the height required is $300 - 144\frac{3}{4}$, or $155\frac{1}{4}$ feet.

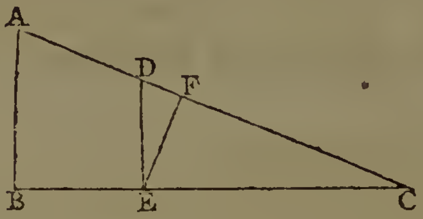
Ex. 2. If a body be projected perpendicularly upwards with a velocity of 80 feet per ſecond, to find it's place at the end of 6″.

The ſpace which would be deſcribed in 6″, with the firſt velocity, is 480 feet, and the ſpace fallen through in the ſame time is $16\frac{1}{12} \times 36$, or 579 feet; therefore the diſtance of the body from the point of projection, at the end of 6″, is 480 – 579, or – 99 feet. The negative ſign ſhews that the body will be below the point of projection (Vid. Alg. Art. 474).

Prop. LXI.

249. *The force which accelerates or retards a body's motion upon an inclined plane, is to the force of gravity, as the height of the plane to it's length.*

Let *AC* be the plane, *BC* it's baſe parallel to the horizon, *AB* it's perpendicular height, *D* the place of a body upon it. From the point *D* draw *DE* parallel to *AB*, and take *DE* to repreſent the force of gravity; from *E* draw *EF* perpendicular to *AC*. Then the whole force *DE* is equivalent to the two *DF*, *FE*, of which *FE* is perpendicular to the plane, and conſequently, is ſupported by the plane's reaction (Art. 116); the other force *DF*, not being affected by the plane, is wholly employed in accelerating or retarding the motion of the body in the direction of the plane; therefore, the accelerating force : the force of gravity :: *DF* : *DE* :: (from the ſimilar △s *DEF*, *ABC*) *AB* : *AC*.



250. Cor. 1. Since the accelerating force, on the ſame plane, is in a given ratio to the force of gravity, it is an uniform force.

251. Cor. 2. If H be the height of an inclined plane, L it's length, and the force of gravity be repreſented by unity, the accelerating force on the inclined plane is repreſented by $\frac{H}{L}$.

For, the accelerating force : the force of gravity (1) :: $H : L$; therefore the accelerating force $= \frac{H}{L}$.

252. Cor. 3. Since $H : L$ :: the ſine of the plane's inclination : the radius, $\frac{H}{L}$, or the accelerating force, varies as the ſine of the plane's inclination to the horizon.

253. Cor. 4. If a body fall down an inclined plane, the velocity V, generated in T'', is ſuch as would carry it uniformly over $\frac{H}{L} \times 2mT$ feet in $1''$; where $m = 16\frac{1}{12}$.

In general, $V \propto FT$ (Art. 228); therefore, the velocity acquired when a body falls by the force of gravity : the velocity acquired on the inclined plane :: the product of the numbers which repreſent the force and time in the former caſe : the product of the numbers which repreſent them in the latter *; alſo, the force of gravity being repreſented by unity, the accelerating force upon the plane is $\frac{H}{L}$, and the velocity generated by

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* Vid. note, page 12.

by the force of gravity in 1″ is $2m$; therefore, $2m : V :: 1 \times 1 : \frac{H}{L} \times T$; and $V = \frac{H}{L} \times 2mT$ *.

Ex. Thus, if the length of an inclined plane be twice as great as it's height, a body which falls down this plane will, in 3″, acquire a velocity of $\frac{1}{2} \times 32\frac{1}{6} \times 3$, or $48\frac{1}{4}$ feet per ſecond.

254. Cor. 5. The ſpace fallen through in T″, from a ſtate of reſt, is $\frac{H}{L} \times mT^2$ feet.

In general, $S \propto FT^2$ (Art. 240); therefore, the ſpace through which a body falls by the action of gravity in 1″ : the ſpace through which it falls down the inclined plane in T″ :: the product of the numbers which repreſent the force and ſquare of the time in the former caſe : the product of the numbers which repreſent them in the latter; or, if S be the ſpace deſcribed upon the plane, $m : S :: 1 \times 1^2 : \frac{H}{L} \times T^2$, and $S = \frac{H}{L} \times mT^2$.

Ex. 1. If $L = 2H$, the ſpace through which a body falls in 3″ is $\frac{1}{2} \times 16\frac{1}{12} \times 9$, or $72\frac{3}{8}$ feet.

Ex. 2. To find the time in which a body will deſcend 12 feet down this plane.

Since

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* In this, and the following articles, the planes are ſuppoſed to be perfectly ſmooth, and the reſiſtance of the air inconſiderable.

Since $S=\frac{H}{L}\times mT^{2}$, $T^{2}=\frac{L\times S}{H\times m}=$ (in this caſe) $\frac{2}{1}\times 12\times\frac{1}{16\frac{1}{12}}=1.49$; and $T=1.2$, nearly.

255. Cor. 6. The ſpace through which a body muſt fall, from a ſtate of reſt, to acquire the velocity V, is $\frac{L}{H}\times\frac{V^{2}}{4m}$ feet.

In general, $S\propto\frac{V^{2}}{F}$ (Art. 240); therefore, the ſpace through which the body falls by the force of gravity : the ſpace through which it falls down the plane :: $\frac{V^{2}}{F}$ in the former caſe : $\frac{V^{2}}{F}$ in the latter; and if m ($16\frac{1}{12}$) be the ſpace fallen through by the action of gravity, $2m$ is the velocity acquired; hence, $m : S :: \frac{4m^{2}}{1} : \frac{L}{H}\times V^{2}$; and $S=\frac{L}{H}\times\frac{V^{2}}{4m}$.

Ex. 1. If $L=2H$, and a body fall from a ſtate of reſt till it has acquired a velocity of 30 feet per ſecond, the ſpace deſcribed is $\frac{2}{1}\times\frac{900}{64\frac{1}{3}}=27.97$ feet, nearly.

Ex. 2. If a body fall 12 feet from a ſtate of reſt down this plane, to find the velocity acquired.

Since $S=\frac{L}{H}\times\frac{V^{2}}{4m}$, we have $V^{2}=4mS\times\frac{H}{L}=$ (in this caſe) $64\frac{1}{3}\times 12\times\frac{1}{2}=386$; hence, $V=19.6$ feet per ſecond, nearly.

Prop. LXII.

256. *The velocity acquired in falling down the whole length of an inclined plane varies as the ſquare root of it's height* *.

In general, when the force is uniform, $V^2 \propto FS$ (Art. 243); in this caſe, $F \propto \frac{H}{L}$, and $S = L$, by the ſuppoſition; therefore $V^2 \propto \frac{H}{L} \times L \propto H$; and $V \propto \sqrt{H}$ (Alg. Art. 202).

257. Cor. 1. When the heights of two inclined planes are equal, the velocities acquired in falling down their whole lengths are equal.

258. Cor. 2. The velocity which a body acquires in falling down the length of an inclined plane is equal to the velocity which it would acquire in falling down it's perpendicular height.

Prop. LXIII.

259. *The time of deſcent down the whole length of an inclined plane varies as the length directly, and as the ſquare root of the height inverſely.*

In general, $S \propto TV$ (Art. 238); therefore, $T \propto \frac{S}{V}$; and in this caſe, $V \propto \sqrt{H}$ (Art. 256); conſequently, $T \propto \frac{S}{\sqrt{H}} \propto \frac{L}{\sqrt{H}}$.

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* Bodies, in this, and the ſubſequent propoſitions, are ſuppoſed to fall from a ſtate of reſt.

260. Cor. 1. If the height, or the laſt acquired velocity be given, $T \propto L$.

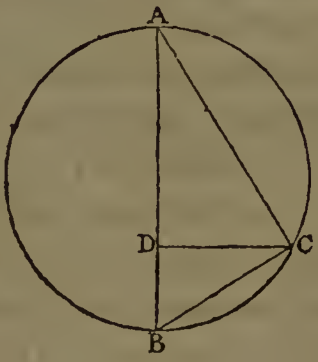
261. Cor. 2. If the inclination be given, or $H \propto L$, then $T^2 \propto \frac{L^2}{L} \propto L$, and $T \propto \sqrt{L}$. That is, the times of deſcent, down planes equally inclined to the horizon, vary as the ſquare roots of their lengths.

262. Cor. 3. The time of deſcent down an inclined plane, is to the time of falling down it's perpendicular height, as the length of the plane to it's height.

PROP. LXIV.

263. *If chords be drawn in a circle from the extremity of that diameter which is perpendicular to the horizon, the velocities which bodies acquire by falling down them are proportional to their lengths; and the times of deſcent are equal.*

Let ACB be the circle, AB it's diameter perpendicular to the horizon; BC a chord drawn from the



extremity B of the diameter; join AC, and draw CD perpendicular to AB, or parallel to the horizon. Then CB may be conſidered as an inclined plane whoſe height is DB, and the velocity acquired in falling down it varies as $\sqrt{DB}$ (Art. 256). Now, from the ſimilar Δs DBC, ABC, $DB : CB :: CB : AB$; therefore $DB = \frac{CB^2}{AB}$, and $\sqrt{DB} = \frac{CB}{\sqrt{AB}}$; conſequently, $V \propto \frac{CB}{\sqrt{AB}}$, and AB is invariable; therefore $V \propto CB$.

Again, $T \propto \frac{S}{V}$ (Art. 239), and in this caſe, CB is the ſpace deſcribed, and it has been proved proportional to the velocity acquired; therefore $T \propto \frac{CB}{CB}$, or the time of deſcent is invariable.

264. Cor. 1. The time of deſcent down any chord CB, is equal to the time of deſcent down the diameter AB.

265. Cor. 2. In the ſame manner, the time of deſcent down AC is equal to the time of deſcent down AB; therefore the time of deſcent down AC is equal to the time of deſcent down CB.

266. Cor. 3. The times of deſcent down the chords thus drawn, in different circles, are proportional to the ſquare roots of the diameters.

For, the times of deſcent down the chords are equal to the times of deſcent down the diameters which are perpendicular to the horizon; and theſe times vary as the ſquare roots of the diameters (Vid. Art. 241).

267. When a body falls freely by the force of gravity,

vity, every particle in it is equally accelerated; that is, every particle deſcends towards the horizon with the ſame velocity; in this deſcent therefore, no rotation will be given to the body. The ſame may be ſaid when a body deſcends along a perfectly ſmooth inclined plane, if that part of the force which acts in a direction perpendicular to the plane (Art. 249) be ſupported; that is, if a perpendicular to the plane, drawn from the center of gravity of the body, cut the plane in a point which is in contact with the body. If this part of the force be not ſuſtained by the plane, the body will partly roll and partly ſlide, till this force is ſuſtained; and afterwards the body will wholly ſlide. When the lateral motion is entirely prevented by the adheſion of the body to the plane, we have before ſeen on what ſuppoſition the body will roll (Art. 186); if the adheſion be not ſufficient to prevent all lateral motion, this body will partly ſlide and partly roll; and to eſtimate the ſpace deſcribed, the time of it's motion, or the velocity acquired, we muſt have recourſe to other principles than thoſe above laid down. On this ſubject the Reader may conſult Mr. VINCE's *Plan of a Courſe of Lectures*, p. 39.

268. When a body falls freely by the force of gravity, or deſcends along a perfectly ſmooth inclined plane, the accelerating force is the ſame whatever be the weight of the body (Arts. 248. 249); conſequently, the moving force, on either ſuppoſition, is proportional to the quantity of matter moved. In all caſes, the accelerating force varies as the moving force directly and the quantity of matter inverſely (Art. 24); and when the moving force and quantity of

matter moved are invariable, the accelerating force is uniform, and it's effects may be eſtimated by the rules laid down in the firſt part of this Section.

Ex. If two bodies, whoſe weights are P and Q, be connected by a ſtring, and hung over a fixed pulley, to find how far the heavier P will deſcend in T''.

The moving force of gravity is proportional to the weight; if therefore P be taken to repreſent the moving force of the former body when it deſcends freely, Q will repreſent the moving force of the latter, and $P-Q$ will repreſent the moving force when the bodies are connected and oppoſe each others motion; hence, neglecting the inertia of the ſtring and pulley, the accelerating force of gravity : the accelerating force in this caſe $:: \frac{P}{P} : \frac{P-Q}{P+Q} :: 1 : \frac{P-Q}{P+Q}$; and, ſince $FT^2 \propto S$, $1 \times 1^2 : \frac{P-Q}{P+Q} \times T^2 :: 16\frac{1}{12} : 16\frac{1}{12} \times \frac{P-Q}{P+Q} \times T^2$, the ſpace required.

⁂

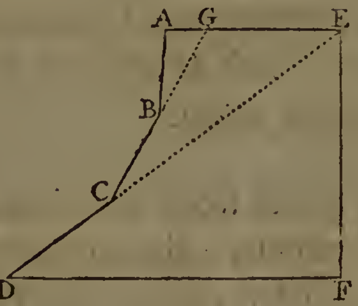
SECTION VIII.

ON THE OSCILLATIONS OF BODIES IN CYCLOIDS AND IN SMALL CIRCULAR ARCS.

Prop. LXV.

269. *IF a body deſcend down a ſyſtem of inclined planes, the velocity acquired, on the ſuppoſition that no motion is loſt in paſſing from one plane to another, is equal to that which would be acquired in falling through the perpendicular height of the ſyſtem.*

Let $ABCD$ be the ſyſtem of planes; draw AE,



DF parallel to the horizon; produce CB, DC till they meet AE in G and E; and draw EF perpendicular to DF. Then the velocity acquired by a body in falling from A to B, is equal to that which it would acquire in falling from G to B, becauſe the planes AB, GB have the ſame perpendicular height (Art. 257); and ſince, by the ſuppoſition, no velocity is loſt in paſſing from one plane to another, the body will begin to deſcend down BC with the ſame velocity, whether it fall down AB or GB; conſequently, the velocity acquired at C will be the ſame on either ſuppoſition. Alſo, the velocity acquired at C is equal to that which would be acquired in falling down EC (Art. 257); and no velocity being loſt at C, the body will begin to deſcend down CD with the ſame velocity, whether it fall from A through B and C to D, or from E to D; and the velocity acquired in falling down ED is equal to the velocity acquired in falling through the perpendicular height EF (Art. 258); therefore, the velocity acquired in falling down the whole ſyſtem, is equal to the velocity acquired in falling through the perpendicular height of the ſyſtem.

Prop. LXVI.

270. *If a body fall from a ſtate of reſt down a curve ſurface which is perfectly ſmooth, the velocity acquired is equal to that which would be acquired in falling through the ſame perpendicular height.*

When a body paſſes from one plane AB to another BC, the whole velocity : the quantity by which the velocity is diminiſhed :: radius : the verſed ſine of the $\angle ABG$ (Art. 221); when, therefore, the angle ABG

is diminiſhed without limit, the velocity loſt is diminiſhed without limit; and if the lengths of the planes as well as their angles of inclination ABG, BCE be continually diminiſhed, the ſyſtem approximates to a curve as it's limit, in which no velocity is loſt; conſequently, the whole velocity acquired is equal to that which a body would acquire in falling through the ſame perpendicular altitude (Art. 269) *.

271. Cor. 1. If a body be projected up a curve, the perpendicular height to which it will riſe is equal to that through which it muſt fall to acquire the velocity of projection.

For

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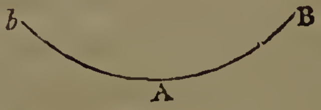
* When the chord of an arc is diminiſhed without limit with reſpect to the diameter, the verſed ſine is diminiſhed without limit with reſpect to the chord; becauſe, the diameter : the chord :: the chord : the verſed ſine; hence, the ratio of the diameter to the verſed ſine, and conſequently, the ratio of the radius to the verſed ſine, is, in this caſe, indefinitely greater than the ratio of the diameter to the chord. Let BC be one of the evaneſcent planes, V the velocity of the deſcending body at B, $V+v$ it's velocity at C; produce CB to G, and let GB be the ſpace through which the body muſt deſcend to acquire the velocity V; then, $V : V+v :: \sqrt{GB} : \sqrt{GB+BC}$; and when $GB : BC ::$ the radius : an evaneſcent chord, $V : V+v :: GB : GB+\frac{BC}{2}$ (Vid. note, page 126.); therefore, $V : v :: GB : \frac{BC}{2} :: 2GB : BC$. Alſo, V : the velocity loſt at B :: radius : the verſed ſine of the angle ABG. Hence it follows, that the ratio of V to the velocity loſt at B, is indefinitely greater than the ratio of V to the velocity acquired in the deſcent from B to C; and conſequently, the velocity loſt at B is indefinitely leſs than the velocity acquired in the deſcent from B to C; in the ſame manner, the velocity loſt at any other plane, is indefinitely leſs than the velocity acquired in the deſcent down that plane; therefore, the velocity loſt in the whole deſcent is indefinitely leſs than the whole velocity acquired.

For the body in it's aſcent will be retarded by the ſame degrees that it was accelerated in it's deſcent.

272. Cor. 2. If *BAb* be a curve in which the loweſt point is *A*, and the parts *AB*, *Ab* are ſimilar and equal, a body in falling down *BA* will acquire a velocity which will carry it to *b*; and ſince the velocities at all equal altitudes in the aſcent and deſcent are equal, the whole time of the aſcent will be equal to the time of deſcent.

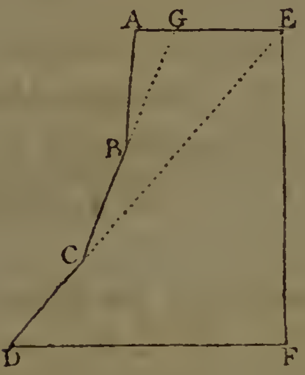


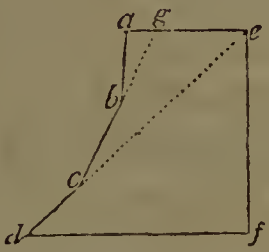
273. Cor. 3. The ſame propoſition is true, if the body be retained in the curve by a ſtring which is in every point perpendicular to it. For the ſtring will now ſuſtain that part of the weight which was before ſuſtained by the curve (Art. 117).

Prop. LXVII.

274. *The times of deſcent down ſimilar ſyſtems of inclined planes, ſimilarly ſituated, are as the ſquare roots of their lengths, on the ſuppoſition that no velocity is loſt in paſſing from one plane to another.*

Let *ABCD*, *abcd*, be two ſimilar ſyſtems of inclined planes, ſimilarly ſituated; that is, let $AB : ab :: BC : bc :: CD : cd$; the ∠s *ABC*, *BCD*, reſpectively equal to the ∠s *abc*, *bcd*; and the planes *AB*, *ab*, equally inclined to the horizon. Complete the figures as in the laſt propoſition; then, ſince $AB : ab ::$ $BC :$





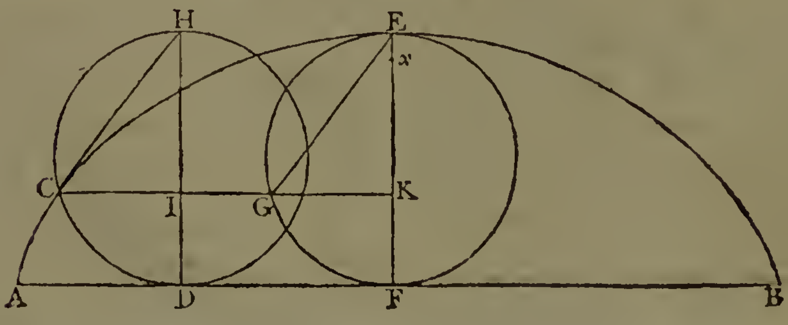
$BC : bc :: CD : cd$, we have, $AB : ab :: AB + BC + CD : ab + bc + cd$ (Alg. Art. 183); and, $\sqrt{AB} : \sqrt{ab} :: \sqrt{AB + BC + CD} : \sqrt{ab + bc + cd}$. Alſo, ſince the ∠s ABC, abc are equal, their ſupplements, the ∠s ABG, abg are equal; and the angles of inclination to the horizon BAG, bag are equal; therefore, the △s ABG, abg are ſimilar, and $AB : BG :: ab : bg$; alt. $AB : ab :: BG : bg :: BC : bc$; conſequently, $BG : bg :: BG + BC\ (GC) : bg + bc\ (gc) :: AB : ab$. In the ſame manner, $ED : ed :: AB : ab$. Then, becauſe the planes AB, ab, are equally inclined to the horizon, the time of deſcent down AB : the time down $ab :: \sqrt{AB} : \sqrt{ab}$ (Art. 261); and if bodies fall down GC, gc, the time down GC : the time down $gc :: \sqrt{GC} : \sqrt{gc} :: \sqrt{AB} : \sqrt{ab}$; alſo, the time down GB : the time down $gb :: \sqrt{GB} : \sqrt{gb} :: \sqrt{AB} : \sqrt{ab}$; hence, the whole time down GC : the whole time down gc :: the time down GB : the time down gb; therefore, the remainder, the time down BC : the remainder, the time down bc, in the ſame ratio, or as $\sqrt{AB} : \sqrt{ab}$ (Euc. 19. 5); and ſince, by the ſuppoſition, no motion

tion is loſt in paſſing from one plane to another, the times of deſcent down BC and bc are the ſame, whether the bodies deſcend from A and a, or from G and g; conſequently, when the bodies deſcend down the ſyſtems, the time down BC : the time down bc :: $\sqrt{AB} : \sqrt{ab}$. In the ſame manner it may be ſhewn that the time down CD : the time down cd :: $\sqrt{AB} : \sqrt{ab}$. Hence, the time down AB: the time down ab :: the time down BC : the time down bc :: the time down CD : the time down cd; therefore, the time down $AB+BC+CD$: the time down $ab+bc+cd$ :: the time down AB : the time down ab :: $\sqrt{AB} : \sqrt{ab}$ (Algebra, Art. 183) :: $\sqrt{AB+BC+CD} : \sqrt{ab+bc+cd}$.

275. Cor. 1. If the lengths of the planes and their angles of inclination ABG, BCE, &c. be continually diminiſhed, the limits, to which theſe ſyſtems approximate, are ſimilar curves, ſimilarly ſituated, in which no velocity is loſt (Art. 270); hence, the whole times of deſcent down theſe curves will be as the ſquare roots of their lengths.

276. Cor. 2. The times of deſcent down ſimilar circular arcs, ſimilarly ſituated, are as the ſquare roots of the arcs, or as the ſquare roots of their radii.

277. Déf. If a circle be made to *roll*, in a given



plane, upon a ſtraight line AB, the point C in the circumference, which was in contact with AB at the beginning of the motion, will, in a revolution of the circle, deſcribe a curve $ACEB$ called a *cycloid.*

The line AB is called the *baſe* of the cycloid.

The circle HCD is called the *generating circle.*

The line FE, which is drawn biſecting AB at right angles, and produced till it meets the curve in E, is called the *axis*, and the point E, the *vertex*, of the cycloid.

278. Cor. 1. The baſe AB is equal to the circumference of the generating circle; and AF to half the circumference.

279. Cor. 2. The axis FE is equal to the diameter of the generating circle.

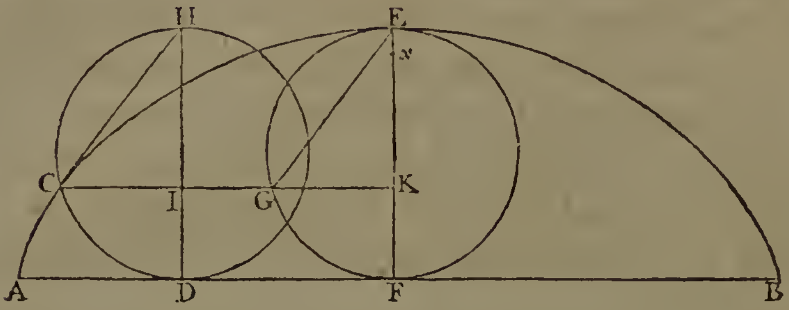
When the generating circle comes to F, draw the diameter Fx, which will be perpendicular to AB (Euc. 18. 3); and becauſe the circle has completed half a revolution, x is the generating point; that is, x is a point in the cycloid, or x coincides with E.

Prop. LXVIII.

280. *If a line CGK, drawn from any point C in the cycloid, parallel to the baſe AB, meet the generating circle, deſcribed upon the axis, in G, the circular arc EG is equal to the right line CG.*

Let the generating circle HCD touch the baſe in D when the generating point is at C; draw DH perpendicular to AB, and it will be the diameter of the circle HCD (Euc. 19. 3), and therefore equal to FE; join CH, GE; and ſince $DH=FE$, and $DI=FK$ (Euc.

34. 1), the remainders IH and KE are equal; conſequently, CI, which is a mean proportional between



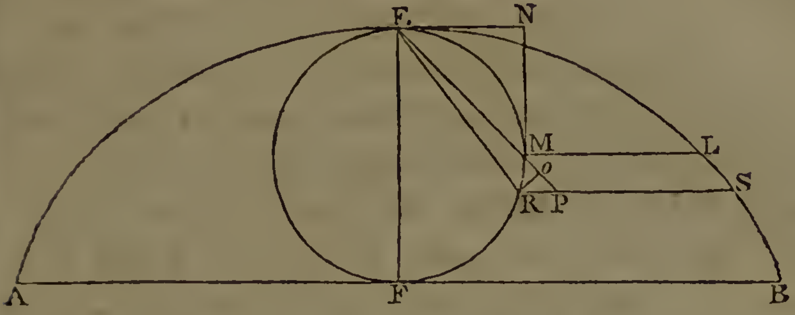
HI and ID (Euc. Cor. 8. 6), is equal to GK, which is a mean proportional between EK and KF; to each of theſe equals add IG, and $CG=IK$. Alſo, CH, which is a mean proportional between IH and HD, is equal to GE, which is a mean proportional between EK and EF; therefore, the arc CH = the arc GE (Euc. 28. 3); and ſince every point in CD has been ſucceſſively in contact with AD, $CD=AD$, and $HCD=AF$ (Art. 278); hence, the arc $CH=DF$; therefore, the arc $EG=DF=IK=CG$.

Prop. LXIX.

281. *If a line* LM, *drawn from* L *parallel to the baſe* AB, *meet the generating circle deſcribed upon the axis in* M, *and* EM *be joined, the tangent to the cycloid at the point* L *is parallel to the chord* EM.

Draw SR parallel and indefinitely near to LM; join EM, RM, SL; produce EM till it meets SR in P; draw EN, MN touching the circle in E and M, and meeting each other in N.

Then ſince RM is ultimately in the direction of the



tangent MN (Newt. Lem. 6), the angles RMP, EMN are equal; and becauſe EN is parallel to RS (Euc. 18. 3), the angles MPR, MEN are equal; therefore, the Δs EMN, RMP are equi-angular, and $EN : MN :: RP : RM$; and ſince $EN = NM$, $RP = RM =$ the arc RM (Newt. Lem. 7). Again, ſince the arc $EMR = RS$ (Art. 280), and $RM = RP$, the remainders, the arc EM and PS are equal; alſo, $ML =$ the arc EM; therefore, $PS = ML$; conſequently, SL is equal, and parallel to PM (Euc. 33. 1); and ſince SL is ultimately in the direction of the tangent at L (Newt. Lem. 6), MP, or EM, is parallel to the tangent at L.

282. Cor. The tangent to the cycloid at B or A, is perpendicular to AB.

Prop. LXX.

283. *The ſame conſtruction being made, the cycloidal arc EL is double of EM the correſponding chord of the generating circle deſcribed upon the axis.*

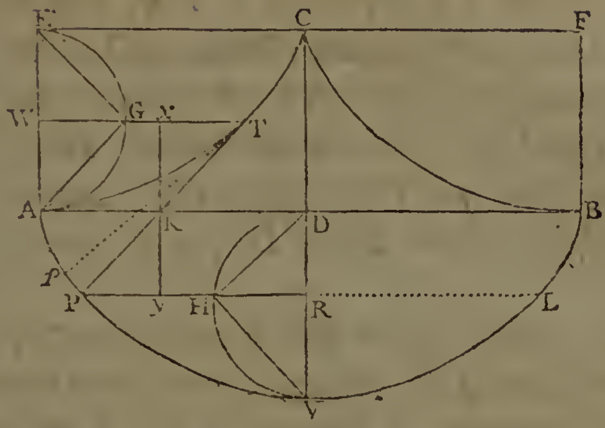
Join ER, and in EP take $Eo=ER$; join Ro. Then, when the arc MR, and conſequently the angle MER, is diminiſhed without limit, the ſum of the angles ERo, EoR approximates to two right angles as it's limit; and theſe angles are equal to each other; therefore, each of them is a right angle; and ſince the angles RMo, RoM are reſpectively equal to RPo, RoP, and Ro is common to the $\triangle$s RoM, RPo, $Mo=oP$, and $MP=2Mo$; alſo, Mo $(ER-EM)$ is the quantity by which the chord EM increaſes, whilſt the cycloidal arc EL increaſes by LS; and it appears from the demonſtration of the laſt propoſition that $MP=LS=$ arc LS (Newt. Lem. 7); therefore, the arc $LS=2Mo$; or, the cycloidal arc EL increaſes twice as faſt as the correſponding chord EM; and they begin together at E; conſequently, the cycloidal arc EL is double of EM, the correſponding chord of the generating circle.

284. Cor. The whole ſemi-cycloidal arc EB is equal to twice the axis EF.

Prop. LXXI.

285. *To make a body oſcillate in a given cycloid.*

Let AVB be the given cycloid, placed with it's vertex downwards, and it's axis DV perpendicular to the horizon. Produce VD to C, making $DC=VD$; complete the rectangles DE, DF; upon AE deſcribe a ſemicircle AGE, and with A as the generating point, and baſe EC, deſcribe a ſemi-cycloid ATC; this will paſs through the point C, becauſe the ſemi-circumference $AGE=DHV=AD=EC$; in the ſame manner, deſcribe an equal ſemi-cycloid between C and B.



Then, if a body P be ſuſpended at C by a ſtring whoſe length is CV or CTA (Art. 284), and made to vibrate between the cycloidal cheeks CA, CB, it will always be found in the cycloid AVB.

Let the ſtring be brought into the ſituation CTP, and ſince it is conſtantly ſtretched by the gravity, and the centrifugal force of P, it will be a tangent to the cycloid at the point T where it leaves the curve. From T and P draw TGW, PHR parallel to AD; join AG, GE, DH, HV; and through K draw xKy perpendicular to TG or PH. Then, ſince the chord AG is parallel to TP (Art. 281), and TG is parallel to AK, the figure GK is a parallelogram, and $AG = TK$, $GT = AK$; and becauſe the length of the ſtring is equal to CTA, and the part CT is common to the ſtring and the cycloidal arc, $TP = AT = 2AG$ (Art. 283) $= 2TK$; or $TK = KP$; hence, the Δs TKx, PKy are ſimilar and equal, and $Kx = Ky$; alſo, $Kx = AW$ and $Ky = DR$; therefore, $AW = DR$; and $AE = DV$; hence, the arc $AG =$ the arc DH; and the $\angle GEA$

$=$ the $\angle HVD$; or, the $\angle GAK =$ the $\angle KDH$ (Euc. 32. 3); conſequently, AG is parallel to DH; and therefore, TP is parallel to DH, and the figure $KPHD$ is a parallelogram; hence, $KD=PH$. Again, ſince the arc $AG=GT$ (Art. 280) $=AK$, the arc $DH =AK$; and the ſemi-circumference $DHV=AD$; therefore, the arc $VH=KD=HP$; that is, P is in the cycloid, whoſe axis is DV, and vertex V (Art. 280).

286. Cor. 1. Since DH is parallel to TP, and VH to the tangent at P, the angle contained between TP and the tangent, or between TP and the curve, is equal to the angle DHV; that is, TP is always perpendicular to the curve.

287. Cor. 2. If Pp be an evaneſcent arc, the perpendiculars to the curve at P and p, ultimately meet in T; and Pp may be conſidered as a circular arc whoſe radius is TP.

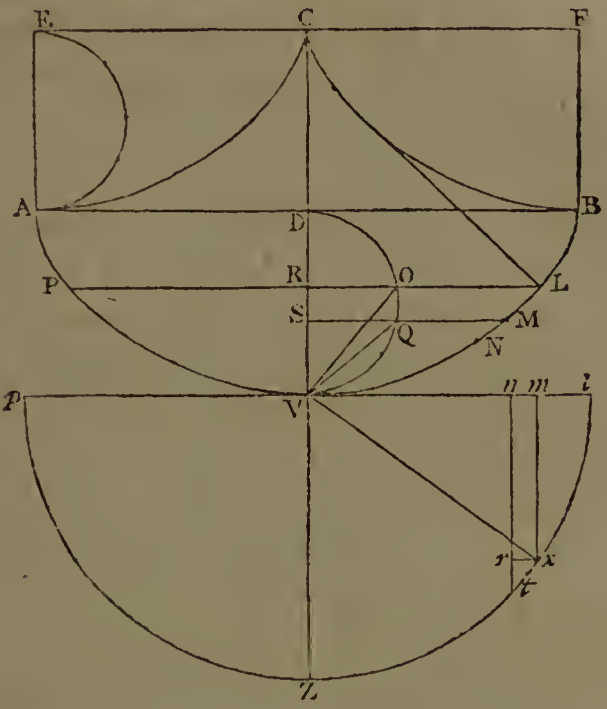
288. Cor. 3. An evaneſcent arc at the vertex of the cycloid may be conſidered as a circular arc whoſe radius is CV.

289. Def. If a body begin to deſcend in a curve, from any point, and again aſcend till it's velocity is deſtroyed (Art. 272), the time in which the motion is performed is called *the time of an oſcillation.*

Prop. LXXII.

290. *If a body, vibrating in the cycloid AVB, begin to deſcend from L, the velocity acquired at any point M varies as $\sqrt{VL^2 - VM^2}$; or, as the right ſine of a circular arc whoſe radius is equal to VL, and verſed ſine to LM.*

From the points L and M, draw LOR, MQS, at right angles to DV, meeting the circle DOV in O



and Q; join OV, QV; with the radius $Vl=VL$, deſcribe the ſemicircle lZp, and take $lm = LM$; draw mx, VZ at right angles to Vl; and join Vx.

The velocity acquired in the deſcent from L to M, is equal to the velocity acquired in falling from R to S (Art. 270); and therefore it varies as $\sqrt{RS}$ (Art. 241); that is, $\propto\sqrt{RV-SV}\propto\sqrt{DV\times RV-DV\times SV}$ (becauſe DV is invariable), $\propto\sqrt{VO^2-VQ^2}\propto\sqrt{4VO^2-4VQ^2}\propto\sqrt{VL^2-VM^2}$ (Art. 283), $\propto\sqrt{Vl^2-Vm^2}\propto\sqrt{Vx^2-Vm^2}\propto\sqrt{mx^2}\propto mx$.

291. Cor. The velocity at M : the velocity at V :: mx : VZ :: mx : Vx.

Prop. LXXIII.

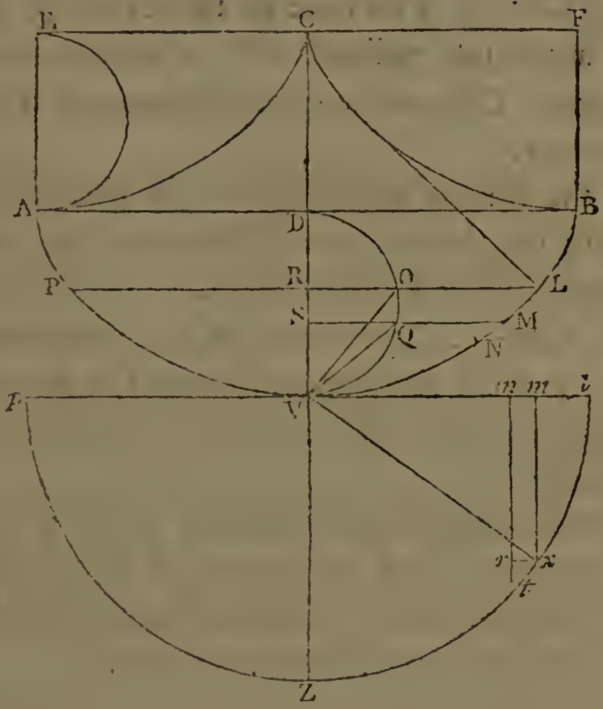
292. *The time of an oſcillation in the arc LVP is equal to the time in which a body would deſcribe the ſemi-circumference lZp with the velocity acquired at V continued uniform.*

Let MN be a very ſmall arc, and take $mn = MN$; draw nt, xr, reſpectively parallel to mx and Vl; and ſuppoſe a body to deſcribe the circumference lZp with the velocity acquired at V continued uniform. Then, when MN is diminiſhed without limit, the velocity with which it is deſcribed : the velocity with which xt is deſcribed :: mx : VZ (Art. 291); therefore the time of deſcribing MN : the time of deſcribing xt :: $\frac{MN}{mx} : \frac{xt}{VZ} :: \frac{xr}{mx} : \frac{xt}{Vx}$ (Art. 15). Now, the Δs Vxm, xrt are ultimately ſimilar, and $Vx : mx :: xt : xr$; therefore $\frac{xr}{mx} = \frac{xt}{Vx}$; conſequently, the time of deſcent down MN, is equal to the time of deſcribing the correſponding circular arc xt with the velocity VZ; and the ſame may be proved of all other correſponding arcs in the cycloid and the circle; therefore, the whole time of an oſcillation is equal to the time of deſcribing the ſemi-circumference lZp, with the velocity acquired at V continued uniform.

Prop. LXXIV.

293. *The time of an oſcillation in a cycloid is to the time of deſcent down it's axis, as the circumference of a circle to it's diameter.*

If a body fall down the chord OV, the velocity acquired at V is equal to the velocity in the cycloid at V (Art. 270); and with this velocity continued uniform, the body would deſcribe $2OV$, or VL, or Vl, in the time of deſcent down OV (Art. 237); that is, in the time of deſcent down DV (Art. 264). It appears then, that the time of an oſcillation is equal to the time of deſcribing lZp with the velocity acquired in the cycloid at V (Art. 292); and that the time of deſcent down the axis DV is equal to the time of deſcribing Vl with the ſame velocity; therefore, the time of an oſcillation : the time of deſcent down the axis :: the time of deſcribing the circumference lZp, with the velocity VZ : the time of deſcribing Vl with the ſame velocity :: lZp : Vl (Art. 13) :: $2lZp$: $2Vl$ :: the circumference of a circle : it's diameter.



294. Cor. 1. The time of an oſcillation in a given cycloid, at a given place, is the ſame, whether the body oſcillate in a greater or a ſmaller arc.

For, the time of an oſcillation bears an invariable ratio to the time of deſcent down the axis, which, in a given cycloid, is given.

295. Cor. 2. The time of an oſcillation in a ſmall circular arc whoſe radius is CV, is to the time of deſcent down $\frac{1}{2}CV$, as the circumference of a circle to it's diameter.

For, the time of an oſcillation in this circular arc is equal to the time of an oſcillation in an equal arc of the cycloid AVB (Art. 288).

296. Cor. 3. The time of an oſcillation in a cycloid, or ſmall circular arc, when the force of gravity is given, varies as the ſquare root of the length of the ſtring.

For, the time of an oſcillation varies as the time of deſcent down half the length of the ſtring; that is, as the ſquare root of half the length of the ſtring, or as the ſquare root of it's whole length.

Ex. 1. To compare the times in which two pendulums vibrate, whoſe lengths are 4 and 9 inches.

Since $T \propto \sqrt{L}$, we have $T : t :: \sqrt{4} : \sqrt{9} :: 2 : 3$.

Ex. 2. If a pendulum, whoſe length is 39.2 inches, vibrate in one ſecond, in what time will a pendulum vibrate whoſe length is L inches?

$\sqrt{39.2} : \sqrt{L} :: 1 : T = \sqrt{\frac{L}{39.2}}$, the time required, in ſeconds.

Ex. 3. To compare the lengths of two pendulums, whoſe times of oſcillation are as 1 to 3.

Since $T \propto \sqrt{L}$, $T^2 \propto L$; therefore, 1 : 9 :: $L : l$.

297. Cor. 4. The number of oſcillations, which a pendulum makes in a given time, at a given place, varies inverſely as the ſquare root of it's length.

Let n be the number of oſcillations, t the time of one oſcillation; then, nt is the whole time, which, by the ſuppoſition, is given; therefore, $n \propto \frac{1}{t}$ (Alg. Art. 206), and $t \propto \sqrt{L}$; conſequently, $n \propto \frac{1}{\sqrt{L}}$.

Ex. 1. If a pendulum, whoſe length is 39.2 inches, vibrate ſeconds, or 60 times in a minute, how often will a pendulum whoſe length is 10 inches vibrate in the ſame time?

Since $n \propto \frac{1}{\sqrt{L}}$, we have $\sqrt{10} : \sqrt{39.2} :: 60 : 60 \times \sqrt{3.92} = 118.8$, nearly, the number of oſcillations required.

Ex. 2. If a pendulum, whoſe length is 39.2 inches, vibrate ſeconds, to find the length of a pendulum which will vibrate double ſeconds, or 30 times in a minute.

Since $n \propto \frac{1}{\sqrt{L}}$, we have, $L \propto \frac{1}{n^2}$; and in this caſe, $\overline{30}|^2 : \overline{60}|^2 :: 39.2 : L = 4 \times 39.2 = 156.8$ inches, the length required.

Ex. 3. To find how much the pendulum of a clock which loſes one ſecond in a minute, ought to be ſhortened.

Since the pendulum vibrates 59 times, whilſt a pendulum of 39.2 inches vibrates 60 times, it's length may be found as in the laſt example; $\overline{59}|^2 : \overline{60}|^2 :: 39.2 : 40.5$, it's length; and it ought to be 39.2 inches; therefore, $40.5 - 39.2$, or 1.3 inches, is the quantity by which it ought to be ſhortened, in order that it may vibrate ſeconds.

298. Cor. 5. If the force of gravity be not given, the time of an oſcillation varies as the ſquare root of the length of the pendulum directly, and as the ſquare root of the force of gravity inverſely.

For, the time of an oſcillation varies as the time of deſcent down half the length of the ſtring; and in general, the time of deſcent through any ſpace $\propto \sqrt{\frac{S}{F}}$ (Art. 240); in this caſe, $S = \frac{1}{2} L$; therefore $S \propto L$, and the time of deſcent $\propto \sqrt{\frac{L}{F}}$; hence T, the time of an oſcillation, $\propto \sqrt{\frac{L}{F}}$.

299. Cor 6. If the length of the pendulum be given, $T \propto \frac{1}{\sqrt{F}}$; and $F \propto \frac{1}{T^2}$.

The time in which a given pendulum vibrates, increaſes as it is carried from a greater latitude on the Earth's ſurface to a leſs; therefore, the force of gravity decreaſes as the latitude decreaſes.

300. Cor. 7. The force of gravity at the Equator : the force of gravity at any propoſed latitude :: the

length of a pendulum which vibrates ſeconds at the Equator : the length of a pendulum which vibrates ſeconds at the propoſed latitude.

For, $T \propto \sqrt{\frac{L}{F}}$; if therefore T be given, $\sqrt{F} \propto \sqrt{L}$, or $F \propto L$.

301. Cor. 8. If the chord BV * be drawn, the time of deſcent down the cycloidal arc BV : the time of deſcent down the chord :: DB : BV.

For, the time of deſcent down the arc BV is equal to half the time of an oſcillation (Art. 272); therefore, the time of deſcent down the arc BV : the time of deſcent down DV :: half the circumference of a circle : it's diameter :: DB : DV; alſo, the time of deſcent down DV : the time of deſcent down the chord BV :: DV : BV; therefore, ex æquo, the time of deſcent down the arc BV : the time of deſcent down the chord :: DB : BV.

PROP. LXXV.

302. *The ſpace through which a body falls by the force of gravity in the time of an oſcillation in a cycloid, or ſmall circular arc, is to half the length of the pendulum, as the ſquare of the circumference of a circle to the ſquare of it's diameter.*

The ſpaces through which bodies fall by the action of the ſame uniform force are as the ſquares of the times (Art. 241); and ſince the time of an oſcillation : the time of deſcent down half the length of the pendulum :: the circumference of a circle : it's diameter, the ſpace fallen through in the time of an oſcillation :

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* The line BV is wanting in the figure.

half the length of the pendulum :: the ſquare of the circumference of a circle : the ſquare of it's diameter.

Ex. To find how far a body will fall by the force of gravity in one ſecond, where the length of the pendulum, which vibrates ſeconds, is 39.2 inches.

The circumference of a circle : it's diameter :: 3.14159 : 1; conſequently, the ſpace fallen through in one ſecond : $\frac{39.2}{2}$ :: $\overline{3.14159}|^2$: 1; hence, the ſpace fallen through is $19.6 \times \overline{3.14159}|^2 = 193$ inches, or $16\frac{1}{12}$ feet, nearly.

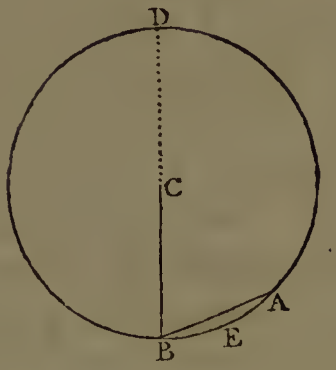
If the arc, in which a body oſcillates, be diminiſhed, the effect of the air's reſiſtance is diminiſhed; and when the arc is very ſmall, this reſiſtance does not ſenſibly diminiſh the time of an oſcillation. By obſerving, therefore, the length of a pendulum which vibrates ſeconds in very ſmall arcs, we determine the ſpace through which a body would fall in vacuo in one ſecond, with ſufficient accuracy for all practical purpoſes.

Prop. LXXVI.

303. *The time of deſcent to the loweſt point in a ſmall circular arc is to the time of deſcent down it's chord, as the circumference of a circle to four times the diameter.*

Let AB be the arc, C it's center, BD the diameter perpendicular to the horizon, T the time of deſcent down the arc AEB, t the time of deſcent down the chord, C the circumference of a circle, D it's diameter. Then, $2T$ is the time of an oſcillation of the pendulum CB; therefore, $2T$: the time down $\frac{CB}{2}$ :: $C : D$ (Art. 295);

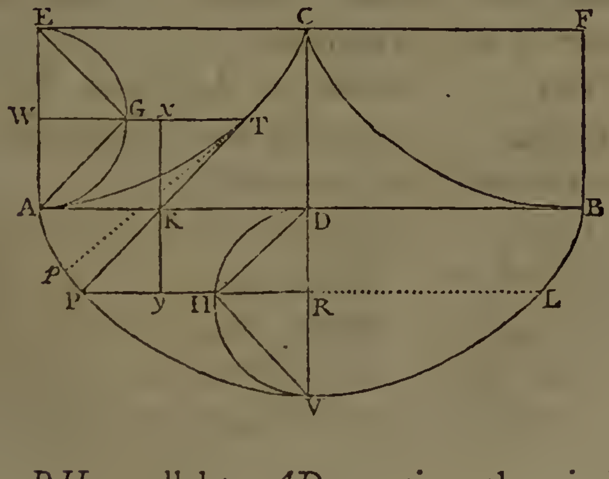
295); and the time down $\frac{CB}{2}$: the time down *DB*, or *AB*, :: 1 : 2 (Art. 241); therefore, $2T$: the time down *AB* (t) :: $C : 2D$, and $T : t :: C : 4D$.



Prop. LXXVII.

304. *The force which accelerates or retards a body's motion in a cycloid varies as the arc intercepted between the body and the loweſt point.*

Let *DV* repreſent the whole force of gravity, from *P* draw *PH* parallel to *AD* meeting the circle *DHV* in *H*; join *DH*, *HV*. Then,



Then, the whole force *DV*, which acts upon the body at *P*, may be reſolved into the two, *DH*, *HV*; of which, *DH* is in the direction of the ſtring, and therefore neither accelerates nor retards the motion of *P*; and *HV* is in the direction of the tangent at *P* (Art. 281), and therefore wholly employed in accelerating or retarding the motion in the curve; conſequently, the force of gravity : the accelerating force :: *DV* : *HV*; and ſince the force of gravity, and *DV* are invariable, the accelerating force $\propto HV \propto 2HV \propto PV$ (Art. 283).

305. Cor. 1. If a body move in any line, and be acted upon by a force which varies as the diſtance from the loweſt point, the motion of this body will be ſimilar to the motion of a body oſcillating in a cycloid.

For, if an arc, meaſured from the vertex of a cycloid, be taken equal to the line, and the accelerating forces, in the line and the cycloid, at theſe equal diſtances from the loweſt points, be equal, they will always be equal, becauſe they vary according to the ſame law; and the bodies, being impelled by equal forces, will be equally accelerated, and deſcribe equal ſpaces in any given time.

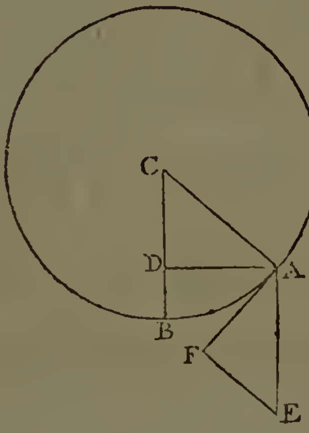
306. Cor. 2. The time of deſcent to the loweſt point in the line will always be the ſame, from whatever place the body begins to fall.

307. Cor. 3. If the diſtance of the body from the loweſt point at the beginning of the deſcent, be made the radius, the velocity acquired will be repreſented by the right ſine, and the time by the arc, whoſe verſed ſine is the ſpace fallen through.

Prop. LXXVIII.

308. *If a body vibrate in a circular arc, the force which accelerates or retards it's motion varies as the ſine of it's diſtance from the loweſt point.*

Let a body oſcillate in a circular arc whoſe radius is AC; from the center C, and A the place of the body, draw CB, AE perpendicular to the horizon, and take AE to repreſent the force of gravity; draw AD perpendicular to CB, and EF perpendicular to AF which is a tangent to the circle at A. Then, the force AE is equivalent to the two AF, FE; of which FE is perpendicular to the tangent AF, or in the direction of the radius CA, and can neither accelerate nor retard the motion of the body; the other, AF, is in the direction of the tangent, and is wholly employed in accelerating or retarding the body's motion; therefore, the force of gravity : the accelerating force :: $AE : AF$, that is, from the ſimilar $\triangle$s AEF, CAD, :: $CA : AD$; and conſequently, the



the accelerating force $= \frac{\text{gravity} \times AD}{AC}$; in which expreſſion, gravity and the radius AC are invariable; therefore, the accelerating force varies as AD.

309. Cor. 1. If the accelerating force were proportional to the arc, the oſcillations, whether in greater or ſmaller arcs, would be performed in equal times (Art. 306); but, ſince the ſine does not increaſe as faſt as the arc, the force in the greater arc is leſs than that which would be ſufficient to make the time of oſcillation equal to the time in the ſmaller arc; therefore, the time of oſcillation in the greater circular arc is greater than in the leſs.

310. Cor. 2. Call F the force in the direction AE, then ſince AC is invariable, the accelerating force in the curve $\propto F \times AD$; and if $F \times AD \propto AB$, or $F \propto \frac{AB}{AD}$, the accelerating force varies as AB, and the times of oſcillation in different arcs are equal (Art. 306).

SCHOLIUM.

SCHOLIUM.

311. In this Section we have conſidered the vibrations of a ſimple pendulum only, or of a ſingle particle of matter, ſuſpended by a ſtring the gravity of which is neglected. The propoſitions are indeed applicable in practice, when the diameter of the body is ſmall with reſpect to the length of the ſtring by which it is ſuſpended, and the weight of the ſtring inconſiderable when compared with the weight of the body. That the concluſions are not ſtrictly true in this caſe, is evident from the conſideration that two particles of matter, at different diſtances from the axis of ſuſpenſion, do not vibrate in the ſame time (Art. 296); and conſequently, that when they are connected together, they affect each other's motion; thus, the time of vibration of the two particles when united, is different from the time in which either would vibrate alone.

The method of determining the time of vibration of a compound pendulum, the Reader will find in the *Principles of Fluxions*, Art. 63; to which place he is alſo referred for the inveſtigation of the rules for determining the centers of *Gyration*, and *Percuſſion*; queſtions properly belonging to Mechanics, but inſerted in that part of the Work, becauſe the rules cannot eaſily be applied to the determination of thoſe points; even in the moſt ſimple caſes, without the aſſiſtance of the fluxional calculus.

312. To avoid the introduction of analytical demonſtrations in ſubjects profeſſedly geometrical, Sir I. Newton and other Writers, have had recourſe to indefinitely ſmall or evaneſcent increments, which continually approximate to the true increments of the quantities whoſe finite values are required. This method may be applied with ſucceſs in all caſes where the *difference* between the *aſſumed* and the *true* increments continually decreaſes, and at length vaniſhes, with reſpect to the increments themſelves; or, which amounts to the ſame thing, when the ratio which the ſum of the differences bears to one of the increments, does not exceed a finite ratio: for, by obſerving the *limit* to which the ſum of the aſſumed increments approaches, when their number is increaſed and their magnitudes are diminiſhed in infinitum, it is evident that the ſum of the real increments is obtained. In the ſame manner, when there are two ranks of quantities, in which the aſſumed increments continually approximate to the real increments, as in the former inſtance, and the limiting ratio of the ſums of the aſſumed increments in theſe caſes, when their numbers are increaſed and their magnitudes diminiſhed without limit, is obtained, the exact ratio of the quantities themſelves is obtained. Theſe propoſitions are laid down by Sir I. Newton in the firſt Section of the Principia, Lem. 3d. and 4th. and the ſame mode of reaſoning has been applied in Art. 292 to compare the time of an oſcillation in the cycloid BVA, with the time of deſcribing the arc lZp with the velocity acquired at V continued uniform. In this Art. it is ſuppoſed that the time of deſcribing MN, with

the *uniform* velocity mx, is the increment of the former time, and that the time of deſcribing xt, the ſide of a triangle *ſimilar* to Vxm, with the velocity VZ, is the increment of the latter; theſe *aſſumed* increments, it is manifeſt, differ from the true increments of the times under conſideration; but when they are diminiſhed without limit, they differ from them by quantities which are evaneſcent with reſpect to the whole increments, and therefore by determining the limiting ratio of the ſums of the aſſumed increments, we obtain the ratio of the actual times of deſcribing the correſponding arcs.

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SECTION IX.

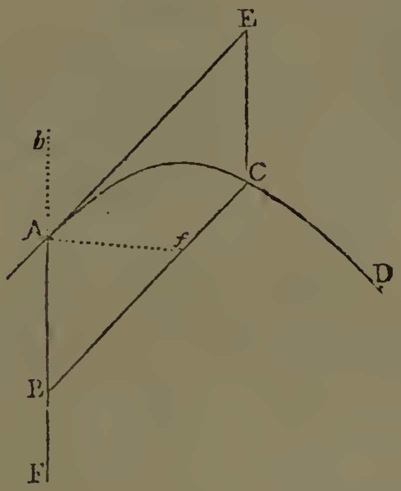
ON THE MOTION OF PROJECTILES.

Prop. LXXIX.

313. *A Body projected in any direction, not perpendicular to the horizon, will deſcribe a parabola, on ſuppoſition that the force of gravity is uniform, and acts in parallel lines, and that the motion is not affected by the reſiſtance of the air.*

Let a body be projected from A in the direction AE, from which point draw ABF perpendicular to the horizon; alſo, let AE be the ſpace over which the velocity of projection would carry the body in any time T, and AB the ſpace through which the force of gravity would cauſe it to deſcend in the ſame time; complete the parallelogram AC; then, in conſequence of the two motions, the body will be found in C at the end of that time. For, the motion in the direction AE

AE can neither accelerate nor retard the approach of the body to the line BC (Art. 29); therefore, at the end of the time T, the body will be in the line BC;



and, by the ſame mode of reaſoning, it appears that it will be in the line EC at the ſame time; conſequently, it will be at C, the point of their interſection, at the end of the time T. Now, ſince AE is the ſpace which would be deſcribed in the time T, with the velocity of projection continued uniform, $AE \propto T$ (Art. 13); and $BC = AE$; therefore, $BC \propto T$, and $BC^2 \propto T^2$. Alſo, ſince AB is the ſpace through which the body would fall by the force of gravity in the time T, $AB \propto T^2$ (Art. 241); hence, $AB \propto BC^2$; and this is the property of a parabola, in which AF is a diameter, and BC an ordinate to the abſciſſa AB.

314. Cor. 1. The axis, and all the diameters of the parabola deſcribed, being parallel to AF, are perpendicular to the horizon.

315. Cor. 2. The direction of projection AE is parallel to the ordinate BC, and therefore it is a tangent to the curve at A.

316. Cor. 3. The time in which a body deſcribes the arc AC is equal to the time in which it would fall from A to B by the force of gravity, or deſcribe AE with the firſt velocity continued uniform.

317. Cor. 4. If the $\angle EAf$ be made equal to the $\angle EAb$, the line Af will paſs through the focus of the parabola deſcribed.

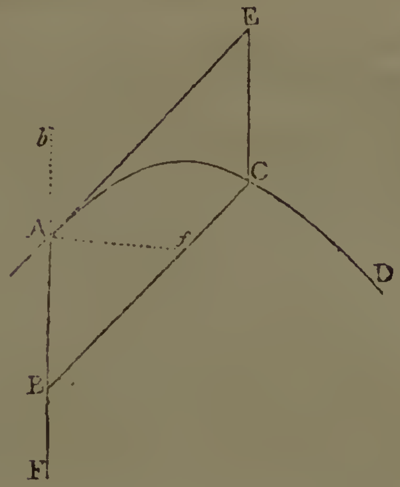
318. Cor. 5. The body will always be found in the plane ACB which paſſes through the direction of projection and the perpendicular to the horizon.

319. Cor. 6. If the motion in the direction AE be produced by the action of an uniform force, $AE \propto T^2 \propto AB$; or $AB \propto BC$; therefore, the locus of the point C is a right line.

Prop. LXXX.

320. *The velocity of the projectile, at any point in the parabola, is ſuch as would be acquired in falling through one fourth part of the parameter belonging to that point.*

Let AB be the ſpace through which a body muſt fall by the force of gravity to acquire the velocity of the projectile at A; and AE the ſpace deſcribed with that velocity continued uniform, in the time of falling through AB; then $2AB = AE$ (Art. 237); and, completing the parallelogram AC, $2AB = BC$; hence, $4AB^2 = BC^2$. Alſo, ſince C is a point in the parabola, and BC an ordinate to the abſciſſa AB (Art.



313), if P be the parameter belonging to the point A, $P \times AB = BC^2 = 4AB^2$; therefore $AB = \frac{1}{4}P$.

321. Cor. 1. If the velocity at A be given, the parameter at that point is the ſame, whatever be the direction of the body's motion.

322. Cor. 2. If AB be the ſpace through which a body muſt fall to acquire the velocity at A, and a circle be deſcribed from the center A with the radius AB, the focus of the parabola deſcribed will lie in the circumference of this circle, whatever be the direction of projection; ſince the diſtance of any point in the parabola from the focus, is one fourth part of the parameter belonging to that point.

323. Cor. 3. The velocity at any point in the parabola varies as the ſquare root of the parameter belonging to that point.

Since the velocity is ſuch as would be acquired in falling through $\frac{1}{4}P$, it varies as $\sqrt{\frac{1}{4}P}$ (Art. 241), or as $\sqrt{P}$.

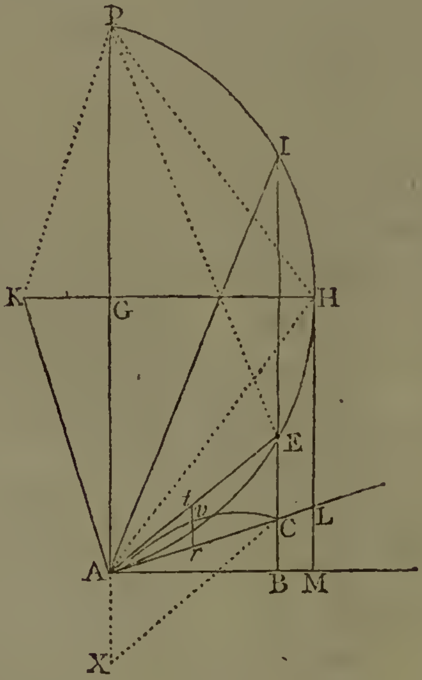
324. Cor. 4. The velocity is the leaſt in the vertex of a parabola; and at equal diſtances from the vertex the velocities are equal.

For, the parameter belonging to the vertex is the leaſt; and the parameters at equal diſtances from the vertex are equal.

Prop. LXXXI.

325. *To find the direction in which a body muſt be projected from a given point, with a given velocity, to hit a given mark.*

Let *A* be the point from which the body is to be projected, *C* the given mark; join *AC*, and from *A* draw



draw *AB* parallel, and *AP* perpendicular to the horizon; take *AP* equal to four times the ſpace through which a body muſt fall to acquire the given velocity of projection, (determined by Art. 248, Caſe 3); then will *AP* be the parameter belonging to the point *A* of the parabola deſcribed (Art. 320). Draw *AK* perpendicular to *AC*; biſect *PA* in *G*, and draw *KGH* perpendicular to *AP*, meeting *AK* in *K*; join *KP*; then, the $\triangle$s *KGP*, *KGA* being ſimilar and equal, $KP = KA$. From *K* as a center with the radius *KA*, or *KP*, deſcribe a circle *AHP*, cutting *KGH* in *H*; through *C* draw *CEI* parallel to *AP*, and cutting the circle in *E* and *I*; join *AE*, *AI*; and if a body be projected, with the given velocity, in the direction *AE* or *AI*, it will hit the mark *C*.

Let the body be projected in the direction *AE*; join *PE*, and complete the parallelogram *AECX*; then *AX* is a diameter of the parabola deſcribed; and *XC*, which is parallel to the tangent *AE*, is in the direction of an ordinate to the abſciſſa *AX*; if then *XC* be the length of the ordinate to this abſciſſa, *C* is a point in the parabola. Now, ſince the $\angle$s *AEC*, *EAP* are alternate angles, and the $\angle$ $EAC =$ the $\angle$ *EPA*, becauſe *AC* is a tangent to the circle at *A* (Euc. 16. 3), the $\triangle$s *EPA*, *EAC* are ſimilar; and $AP : AE :: AE : EC$; or, by ſubſtituting for *AE* and *EC* their equals *XC* and *AX*, $AP : XC :: XC : AX$; that is, *XC* is a mean proportional between the parameter and the abſciſſa; and therefore it is the ordinate belonging to that abſciſſa; hence *C* is a point in the parabola which the body deſcribes.

In the ſame manner it may be ſhewn that, if the body be projected with the ſame velocity in the direction *AI*, it will hit the mark *C*.

326. Cor. 1. Join *AH*, *HP*; then the ∠ *HAP* = the ∠ *HPA* = the ∠ *HAC*.

327. Cor. 2. Becauſe *KH* is drawn through the center of the circle perpendicular to the chord *EI*, it biſects it, and conſequently it biſects the arc *IHE* (Euc. 30. 3); therefore, the ∠ *IAH* = the ∠ *HAE*. That is, the two directions *AE*, *AI* make equal angles with *AH*, which biſects the angle *PAC*.

328. Cor. 3. Draw *HLM* touching the circle in *H*; then, when the point *C* coincides with *L*, the two directions *AE*, *AI*, coincide with *AH*.

329. Cor. 4. If the point *C* be taken in the plane *AL*, beyond *L*, the line *CEI* will not meet the circle. In this caſe, the velocity of projection is not ſufficient to carry the body to the diſtance *AC*.

330. Cor. 5. Biſect *AC* in *r*, and draw *tvr* parallel to *AP*; then *tvr* is in the direction of a diameter, to which *AC* is a double ordinate. Alſo, *rt* is the ſubtangent; and if *rt* be biſected in *v*, this point is in the parabola.

331. Cor. 6. A tangent to the parabola at *v* is parallel to the ordinate *AC*; therefore, *v* is the point in the parabola which is at the greateſt diſtance from *AC*.*

332. Cor. 7. The greateſt height of the projectile above the plane, meaſured in the direction of gravity, is $\frac{1}{4}EC$. For, $rv = \frac{1}{2}rt$, and $rt = \frac{1}{2}EC$; therefore, $rv = \frac{1}{4}EC$.

PROP.

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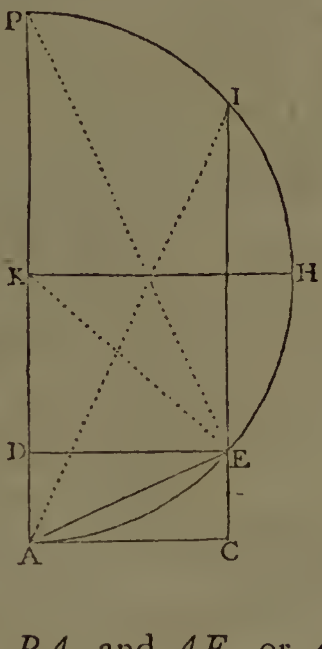
* The properties of the parabola here referred to, may be found in any Treatiſe of Conic Sections.

Prop. LXXXII.

333. *Having given the velocity and direction of projection, to find where the body will ſtrike the horizontal plane which paſſes through the point of projection.*

Let AC (Art. 325) coincide with the horizontal line AB; then AK coincides with AG; and PHA is a ſemicircle; alſo, HAC is an angle of 45°, and AE, AI, are equally inclined to AH.

From the laſt propoſition it appears, that if the velocity of projection be ſuch as would be acquired in falling through $\frac{1}{4}PA$, and AE, or AI, be the direction of projection, the range is AC. From E draw ED parallel to AC, or perpendicular to AP; join EK; then ED, or it's equal AC, is the ſine of the $\angle EKA$ to the radius KA; and the $\angle EKA = 2\angle EPA = 2\angle EAC$; therefore,



therefore, AC is the ſine of $2\angle EAC$ to the radius KA; and the ſine of a given angle is proportional to the radius; conſequently, rad. : KA :: ſin. $2\angle EAC$: AC; hence, $AC = \frac{\text{ſin. } 2\angle EAC \times KA}{\text{rad.}}$. If V be taken to repreſent the velocity of projection, P the parameter AP, and $m = 16\frac{1}{12}$, then $AC = \frac{\text{ſin. } 2\angle EAC \times P}{2\text{ rad.}} = \frac{\text{ſin. } 2\angle EAC \times V^2}{2\text{ rad.} \times m}$ (Art. 248).

In the ſame manner, if AI be the direction of projection, $AC = \frac{\text{ſin. } 2\angle IAC \times P}{2\text{ rad.}} = \frac{\text{ſin. } 2\angle IAC \times V^2}{2\text{ rad.} \times m.}$.

334. Cor. 1. Hence $AC \propto \text{ſin. } 2\angle EAC \times V^2$.

335. Cor. 2. If the velocity of projection be invariable, the horizontal range varies as ſin. $2\angle EAC$.

336. Cor. 3. The range is the greateſt when ſin. $2\angle EAC$ is the greateſt; that is, when the $\angle EAC$ is 45°. In this caſe, $AC = \frac{\text{rad.} \times P}{2\text{ rad.}} = \frac{1}{2}P$.

337. Cor. 4. If the $\angle EAC$ be 15°, or 75°, ſin. $2\angle EAC = \frac{1}{2}$ rad. and $AC = \frac{1}{4}P$.

338. Cor. 5. If the range and the parameter be given, the angle of elevation may be found.

For, $AC = \frac{\text{ſin. } 2\angle EAC \times P}{2\text{ rad.}}$; therefore, ſin. $2\angle EAC = \frac{2AC \times \text{rad.}}{P}$.

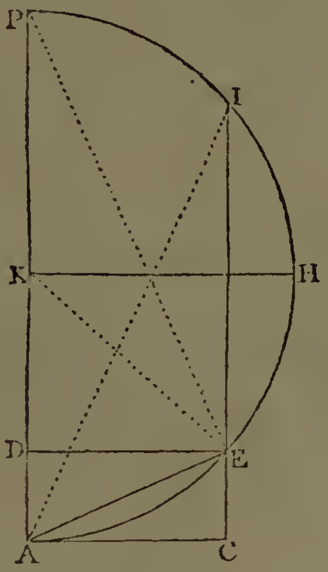
339. Cor. 6. If AC and the $\angle EAC$ be known, $P = \frac{2AC \times \text{rad.}}{\text{ſin. } 2\angle EAC}$; and $V^2 = \frac{2\text{ rad.} \times mAC}{\text{ſin. } 2\angle EAC}$.

PROP.

Prop. LXXXIII.

340. *The ſame things being given, to find the time of flight.*

The time in which the velocity of projection, V, would be acquired, or the time of deſcent down $\frac{1}{4} PA$, is $\frac{V}{2m}''$ (Art. 248); hence the double of this time, or the time of deſcent down PA, is $\frac{V}{m}''$ (Art. 241). Let T be the time of deſcent down PA, t the time of deſcent down EC, or the time of flight (Art. 316); then $T^2 : t^2 :: PA : EC$; and from the ſim. $\triangle$s PAE, AEC, $PA : AE :: AE : EC$; hence $PA^2 : AE^2 :: PA : EC :: T^2 : t^2$, and $PA : AE :: T : t$. Alſo, $PA : AE ::$ rad. : ſin. $\angle EPA ::$ rad. : ſin. $\angle EAC$; therefore $T : t ::$ rad. : ſin. $\angle EAC$, and $t = \frac{\text{ſin.} \angle EAC \times T}{\text{rad.}}$ $= \frac{\text{ſin.} \angle EAC \times V}{\text{rad.} \times m}$



341. Cor. 1. If the velocity of projection be invariable, the time of flight $\propto$ ſin. $\angle EAC$.

342. Cor. 2. Hence, the time of flight is the greateſt, when ſin. $\angle EAC$ is the greateſt. In this caſe, the time becomes $\frac{\text{rad.} \times T}{\text{rad.}}$, or T; that is, the greateſt time of flight is equal to the time of deſcent down the parameter.

Prop. LXXXIV.

343. *The ſame things being given, to find the greateſt height to which the projectile riſes above the horizontal plane.*

The greateſt height is $\frac{1}{4} EC$, or $\frac{1}{4} AD$; and AD is the verſed ſine of the $\angle AKE$, or $2 \angle EAC$, to the radius AK; conſequently, ſince the verſed ſine of a given angle varies as the radius, rad. : AK $(\frac{1}{2} P)$ :: the verſed ſine of $2 \angle EAC$: AD. Hence, $AD = \frac{\text{ver.ſin.}\, 2 \angle EAC \times P}{2\ \text{rad.}}$; and $\frac{1}{4} AD$, the greateſt height, $= \frac{\text{ver.ſin.}\, 2 \angle EAC \times P}{8\ \text{rad.}} = \frac{\text{ver.ſin.}\, 2 \angle EAC \times V^2}{8\ \text{rad.} \times m}$.

344. Cor. 1. The greateſt height $\propto$ ver. ſin. $2 \angle EAC \times V^2$; and when V is given, the greateſt height $\propto$ ver. ſin. $2 \angle EAC$.

345. Cor. 2. The verſed ſine of an arc varies as the ſquare of the ſine of half the arc; therefore the greateſt height $\propto \overline{\text{ſin.}\ \angle EAC)}^2 \times V^2$.

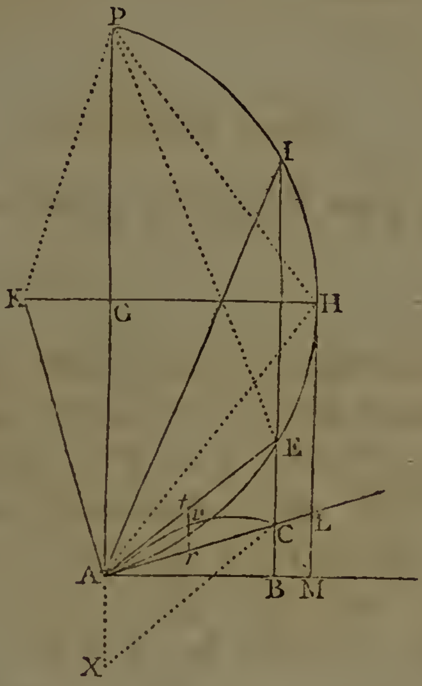
346. Cor. 3. A body, projected with a given velocity, will riſe to the greateſt height above the horizontal plane, when the angle of elevation EAC is a right angle. In this caſe, ver. ſin. $2 \angle EAC = 2$ rad. and the greateſt altitude is $\frac{2\ \text{rad.} \times P}{8\ \text{rad.}} = \frac{P}{4}$.

Prop.

Prop. LXXXV.

347. *The velocity and direction of projection being given, to find where the body will ſtrike a given inclined plane which paſſes through the point of projection.*

It appears from Art. 325 that if a body be projected from A, in the direction AE, with the velocity acquired in falling down $\frac{1}{4}PA$, it will ſtrike the



plane AC in the point C. Let I be the angle of inclination CAB; E the angle of elevation EAC; Z the angle EAP. Then, in the triangle EAP, $AE : AP :: \text{ſin.} \angle EPA : \text{ſin.} \angle AEP$; and the $\angle EPA =$ the $\angle EAC = E$; alſo the $\angle AEP =$ the $\angle ECA =$ the ſupplement of the $\angle ACB$; hence, $AE : AP :: \text{ſin.}\ E : \text{coſ.}\ I$; therefore, $AE = \dfrac{\text{ſin.}\ E \times AP}{\text{coſ.}\ I}$.

Again, in the $\triangle EAC$, $AC : AE :: \text{ſin.} \angle AEC$ (ſin. Z) : ſin. $\angle ACE$ (coſ. I); therefore, $AC = \frac{\text{ſin.}\, Z \times AE}{\text{coſ.}\, I}$; and by ſubſtituting for AE it's value $\frac{\text{ſin.}\, E \times AP}{\text{coſ.}\, I}$, $AC = \frac{\text{ſin.}\, E \times \text{ſin.}\, Z \times AP}{\overline{\text{coſ.}\, I}|^2} = \frac{\text{ſin.}\, E \times \text{ſin.}\, Z \times V^2}{\overline{\text{coſ.}\, I}|^2 \times m}$.

348. Cor. Hence, $AC \propto \frac{\text{ſin.}\, E \times \text{ſin.}\, Z \times V^2}{\overline{\text{coſ.}\, I}|^2}$.

Prop. LXXXVI.

349. *The ſame things being given, to find the time of flight.*

Let T be the time of deſcent down PA, t the time of deſcent down EC or the time of flight; then, as in Art. 340, $T^2 : t^2 :: PA : EC$; and ſince, in the ſimilar $\triangle$s PAE, AEC, $PA : AE :: AE : EC$, we have $PA^2 : AE^2 :: PA : EC :: T^2 : t^2$; and $PA : AE :: T : t$; but $PA : AE :: \text{ſin.} \angle PEA : \text{ſin.} \angle EPA :: \text{ſin.} \angle ECA : \text{ſin.} \angle EAC :: \text{coſ.}\, I : \text{ſin.}\, E$; therefore, $T : t :: \text{coſ.}\, I : \text{ſin.}\, E$, and $t = \frac{\text{ſin.}\, E \times T}{\text{coſ.}\, I} = \frac{\text{ſin.}\, E \times V}{\text{coſ.}\, I \times m}$.

350. Cor. Hence $t \propto \frac{\text{ſin.}\, E \times V}{\text{coſ.}\, I}$; and if V be invariable, $t \propto \frac{\text{ſin.}\, E}{\text{coſ.}\, I}$.

PROP. LXXXVII.

351. *The ſame things being given, to find the greateſt height of the projectile above the plane AC, meaſured in the direction of gravity.*

The greateſt height is $\frac{1}{4}EC$ (Art. 332); and in the triangle AEC, $EC : AE :: \text{ſin. } E : \text{coſ. } I$; therefore, $EC = \frac{\text{ſin. } E \times AE}{\text{coſ. } I}$; and, by ſubſtituting for AE it's value $\frac{\text{ſin. } E \times AP}{\text{coſ. } I}$ (Art. 347), we have $EC = \frac{\overline{\text{ſin. } E}|^2 \times AP}{\overline{\text{coſ. } I}|^2}$; and $\frac{1}{4}EC = \frac{\overline{\text{ſin. } E}|^2 \times AP}{4\,\overline{\text{coſ. } I}|^2} = \frac{\overline{\text{ſin. } E}|^2 \times V^2}{\overline{\text{coſ. } I}|^2 \times 4m}$, the greateſt height required.

352. Cor. The greateſt height varies as $\frac{\overline{\text{ſin. } E}|^2 \times V^2}{\overline{\text{coſ. } I}|^2}$.

* * *

SCHOLIUM.

353. The theory of the motion of projectiles, given in this ſection, depends upon three ſuppoſitions, which are all inaccurate; 1ſt. that the force of gravity, in every point of the curve deſcribed, is the ſame; 2d. that it acts in parallel lines; 3d. that the motion is performed in a non-reſiſting medium. The two former of theſe, indeed, differ inſenſibly from the truth. The force of gravity without the Earth's ſurface varies inverſely as the ſquare of the diſtance from the center; and the altitude to which we can project a body from the ſurface is ſo ſmall, that the variation of the force, ariſing from the alteration of the diſtance from the center of the Earth, may ſafely be neglected. The direction of the force is every where perpendicular to the horizon; and if perpendiculars be thus drawn, from any two points in the curve which we can cauſe a body to deſcribe, they may be conſidered as parallel, ſince they only meet at, or nearly at, the center of the Earth. Even the reſiſtance of the air does not materially affect the motions of heavy bodies, when they are projected with ſmall velocities. In other caſes, however, this reſiſtance is ſo great as to render the concluſions drawn from the theory almoſt entirely inapplicable in practice. From experiments made to determine the motions of cannon

balls, it appears that when the initial velocity is conſiderable, the air's reſiſtance is 20 or 30 times as great as the weight of the ball; and that the horizontal range is often not $\frac{1}{20}$ part of that which the preceding theory gives. It appears alſo, that when the angle of elevation is given, the horizontal range varies nearly as the ſquare root of the velocity of projection; and the time of flight as the range; whereas, according to the theory, the time varies as the velocity, and the range as the ſquare of the velocity of projection (Arts. 340. 334). Theſe experiments, made with great care, and by men of eminent abilities, ſhew how little the parabolic theory is to be depended upon in determining the motions of military projectiles. See ROBINS's *New Theory of Gunnery*, and HUTTON's *Mathematical Dictionary*, article *Gunnery*.

Beſides diminiſhing the velocity of the projectile, the air's reſiſtance will alſo change it's direction, whenever the body has a rotatory motion about an axis which does not coincide with the direction in which it is moving. For the velocity of that ſide of the body, on which the rotatory and progreſſive motions conſpire, is greater than the velocity of the other ſide, where they are contrary to each other; and therefore the reſiſtance of the air, which increaſes with the velocity, will be greater in the former caſe than in the latter, and cauſe the body to deviate from the line of it's motion; this deviation will alſo be from the *plane* of the firſt motion, unleſs the axis of rotation be perpendicular to that plane.

Upon this principle Sir I. NEWTON explains the irregular motion of a tennis ball *, and the ſame cauſe has been aſſigned by Mr. ROBINS for the deviation of a bullet from the vertical plane *. Mr. EULER, indeed, in his remarks on the *New Theory of Gunnery*, contends that the reſiſtance of the air can neither be increaſed nor diminiſhed by the rotation of the ball; becauſe ſuch a motion can produce no effect but in the direction of a tangent to the ſurface of the revolving body; and the tangential force, he affirms, is almoſt entirely loſt. In this inſtance, the learned Writer ſeems to have been miſled by the common theory of reſiſtances, according to which the tangential force produces no effect; whereas, from experiments lately made, with a view to aſcertain the quantity and laws of the air's reſiſtance, it appears that every theory which neglects the tangential force muſt be erroneous.

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* Phil. Tranſ. Vol VI. p. 3078. MACLAURIN's NEWTON, p. 120.

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* Tracts, Vol. I. p. 151, 198, 214;

THE END OF PART I. VOL III.

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ERRATA.

Page 49, l. 18, and p. 93, l. 23, 26, for friction read *adheſion.*

104, after 176. add *Cor.* 4.; and l. 8, for Cor. read *Cor.* 5.

134, note, for volocity read *velocity.*

ERRATA IN THE FLUXIONS.

Page 11, l. 16, for zxy read $zx\dot{y}$.

47, l. 11, after the firſt Term, the Index of x muſt be annexed to a, and that of a to x.

79, l. 20, dele — before $\frac{px^3\dot{x}}{-x+a}$.

83, l. 4 and 7, for xy read vz.

143, l. 17, for $6x^3$ read $6x^2$.

169, l. 6, for 127 read 130.

180, l. 10, for c read $\dot{c}$.

191, l. 18, for $n-1$ read $n-2$.

192, l. 1, for $n-1$ read $n-2$.

209, in the ſecond figure an *O* is wanting in the center:

THE
PRINCIPLES OF HYDROSTATICS:
DESIGNED
FOR THE USE OF STUDENTS
IN THE
UNIVERSITY.

By the Rev. S. VINCE, A.M. F.R.S.
PLUMIAN PROFESSOR OF ASTRONOMY AND EXPERIMENTAL
PHILOSOPHY.

VOL. III. PART II.

CAMBRIDGE,
PRINTED BY J. BURGES, PRINTER TO THE UNIVERSITY;
AND SOLD BY J. DEIGHTON, J. NICHOLSON, AND W. H. LUNN,
CAMBRIDGE; F. WINGRAVE, H. GARDNER, AND
P. EMSLEY, IN THE STRAND; B. & J. WHITE,
FLEETSTREET; F. & C. RIVINGTON, AND
G. & T. WILKIE, IN ST. PAUL'S
CHURCHYARD, LONDON.

MDCCXCVI.

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THE

PRINCIPLES OF HYDROSTATICS.

DEFINITIONS.

Art. 1. THE ſcience which treats of the nature and properties of Fluids has been uſually divided into the following branches; *Hydroſtatics*, which compriſes the doctrine of the equilibrium of *non-elaſtic* fluids, as water, mercury, &c.; *Hydraulics*, which relates to the motion of thoſe fluids; and *Pneumatics*, which treats on the properties of the different kinds of airs. But theſe are now all included under the general term *Hydroſtatics*.

2. A *Fluid* is a body whoſe parts are put in motion one amongſt another by any force impreſſed; and which, when the impreſſed force is removed, reſtores itſelf to it's former ſtate *.

Fluids may be divided into *elaſtic* and *non-elaſtic*.

---

* By it's *former ſtate*, we do not mean that every particle is re-inſtated in it's former ſituation, but that the whole body recovers it's former dimenſions and figure.

An *elaſtic* fluid is one, whoſe dimenſions are diminiſhed by increaſing the preſſure, and increaſed by diminiſhing the preſſure upon it; of which deſcription are the different kinds of airs. A *non-elaſtic* fluid is one, whoſe dimenſions are not, at leaſt as to ſenſe, affected by any increaſe of preſſure, as water, mercury, &c. As many bodies, by cold, from a ſtate of fluidity become ſolids, ſuch bodies are fluids ſo long as their ſurfaces, when diſturbed, will reſtore themſelves to their former poſition. The definition ſuppoſes a *partial* preſſure; for if the fluid be incompreſſible, under an equal and general preſſure none of the parts will be moved. Different fluids have different degrees of fluidity, according to the facility with which the particles are moved one amongſt another. Water and mercury are the moſt perfect non-elaſtic fluids. Many fluids have a very ſenſible degree of tenacity, and are therefore called imperfect fluids. Beſides the fluids which come under this definition, there are others, as the electric and magnetic fluid, light, and fire, according to the opinion of ſome, &c. but theſe are not the objects of Hydroſtatics.

3. The *ſpecific gravity* of a body is it's weight, compared with the weight of another body whoſe magnitude is the ſame.

4. The *denſity* of a body is as the quantity of matter contained in a given ſpace, and therefore (Mech. Art. 26) in proportion to it's weight, when the magnitude is the ſame.

5. Cor. Hence the *ſpecific gravity* of a body is in proportion to it's *denſity*.

A cubic inch of pure mercury is about fourteen times heavier than a cubic inch of water; the ſpecific gravity and denſity of the former are therefore about fourteen times that of the latter. As the weight of a body is in proportion to it's quantity of matter (Mechanics, Art. 26), the ſpecific gravity and denſity of a body are alſo in proportion to it's quantity of matter, when the magnitude is the ſame.

6. If the magnitude of a body be increaſed, the denſity remaining the ſame, the quantity of matter, and conſequently the weight, will be increaſed in the ſame proportion. Hence if the magnitude M, and denſity D both vary, the quantity of matter, and conſequently the weight of the body, will vary as $M \times D$, by the compoſition of ratios.

7. We know ſo little of the nature and conſtitution of fluids, that the application of the general principles of motion to the inveſtigation of the effects produced by their action, is ſubject to great uncertainty. That the different kinds of airs are conſtituted of particles endued with repulſive powers, is manifeſt from their expanſion, when the force with which they are compreſſed is removed. The particles being kept at a diſtance by their mutual repulſion, it is eaſiy to conceive that they may move very freely amongſt each other, and that this motion may take place in all directions, each particle exerting it's repulſive power equally on all ſides. Thus far we are acquainted with the conſtitution of theſe fluids; but with what degree of facility the particles move, and how this may be affected under different degrees of compreſſion, are circumſtances of which we are totally ignorant. With reſpect to the nature of thoſe fluids which are denominated liquids, we are ſtill leſs acquainted. If we ſuppoſe their particles to be in contact, it is very diffi-

cult to conceive how they can move amongſt each other with ſuch extreme facility, and produce effects in directions oppoſite to the impreſſed force, without any ſenſible loſs of motion. To account for this, the particles have, by ſome, been ſuppoſed to be perfectly ſmooth and ſpherical. If we were to admit this ſuppoſition, it would yet remain to be ſhown how it would ſolve all the phænomena, for it is by no means ſelf-evident that it would. If the particles be not in contact, they muſt be kept at a diſtance by ſome repulſive power. But it is manifeſt that theſe particles attract each other, from the drops of all perfect fluids endeavouring to form themſelves into ſpheres. We muſt therefore admit in this caſe both powers, and that where one power ends the other begins, agreeable to Sir I. Newton's * idea of what takes place, not only in reſpect to the conſtituent particles of bodies, but to the bodies themſelves. The incompreſſibility of liquids (for I know no deciſive experiments which have proved them to be compreſſible) ſeems moſt to favour the former ſuppoſition, unleſs we admit, in the latter hypotheſis, that the repulſive force is greater than any human power which can be applied. The expanſion of water by heat, and the poſſibility of actually converting it into two permanent elaſtic fluids, according to ſome late experiments, ſeem to prove that a repulſive power exiſts between the particles, for it is hard to conceive that heat can actually create any ſuch new powers, or that it can of itſelf produce any ſuch effects.

A fluid being compoſed of an indefinite number of corpuſcles, we muſt conſider it's action, either as the joint

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* See his Optics, Que. 31.

joint action of all the corpuſcles, eſtimated as ſo many diſtinct bodies, or we muſt conſider the action of the whole as a maſs, or as one body. In the former caſe, the motion of the particles being ſubject to no regularity, or at leaſt to none that can be diſcovered by any experiments, it is impoſſible, from this conſideration to compute the effect; for no calculation of effects can be applied, when produced by cauſes which are ſubject to no law. And in the latter caſe, the effects of the action of one fluid upon another differ ſo much, in many reſpects, from what would be it's action as a ſolid body, that a computation of it's effects can by no means be deduced from the ſame principles. In mechanics, no equilibrium can take place between two bodies of different weights, unleſs the lighter acts at ſome mechanical advantage; but in hydroſtatics, a very ſmall weight of fluid may, without it's acting at any mechanical advantage whatever, be made to balance a weight of any magnitude. In mechanics, bodies act only in the direction of gravity; but the property which fluids have of acting equally in all directions, produces effects of ſuch an extraordinary nature, as to ſurpaſs the power of inveſtigation. The indefinitely ſmall corpuſcles of which a fluid is compoſed, probably poſſeſs the ſame powers, and would be ſubject to the ſame laws of motion, as bodies of finite magnitudes, could any two of them act upon each other by contact; but this is a circumſtance which certainly never takes place in any of the aerial fluids, and probably not in any liquids. Under the circumſtances, therefore, of an indefinite number of bodies acting upon each other by repulſive powers, or by abſolute contact, under the uncertainty of the friction which may take place, and of what variation of effects may be produced by different degrees of compreſſion, the concluſions deduced from any *theory* muſt be ſubject to conſiderable errors, except from *that* which is founded upon ſuch experiments, as include in them the conſequences of thoſe principles which are liable to any degree of uncertainty.

Sir I. NEWTON ſeems to have been well aware of all theſe difficulties, and therefore, in his *Principia*, he has deduced his laws of reſiſtance, and the principles upon which the times of emptying veſſels are founded, entirely from experiment. He was too cautious to truſt to theory alone, under all the uncertainties to which, he appears to have been ſenſible, it muſt be ſubject. He had, in a preceding part of that great work, deduced the general principles of motion, and applied them to the ſolution of problems which had never before been attempted; but when he came to treat of fluids, he ſaw it was neceſſary to eſtabliſh his principles upon experiments; principles, not indeed mathematically true, like his general principles of motion before delivered, but, under certain limitations, ſufficiently accurate for practical purpoſes. The principles therefore upon which we reaſon in this branch of philoſophy, muſt be conſidered as depending upon direct experiments adapted to the particular caſes, rather than upon the general principles of equilibrium and motion, as applied in Mechanics.

SECTION I.

ON THE PRESSURE OF NON-ELASTIC FLUIDS.

PROP. I.

FLUIDS preſs equally in all directions.

8. For if any liquid be put into a veſſel, and glaſs tubes, ſtraight, or bent at any angle, be put into the fluid, the fluid will riſe in all the tubes to the height of the ſurface of the liquid in the veſſel. Alſo, a veſſel is found to empty itſelf in the ſame time through an hole at the ſame depth, whether the fluid ſpouts downwards, horizontally, upwards, or in any other direction.

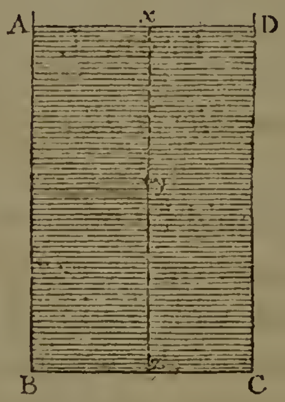
9. That the preſſure upwards is equal to the preſſure downwards, is one of the moſt extraordinary properties of fluids, and can be conceived to ariſe only from the perfect freedom with which the particles

move amongſt each other, which freedom of motion is, probably, owing to the particles being kept at a diſtance by a repulſive power reſiding in them. This is one remarkable difference between fluids and ſolids, ſolids preſſing only downwards in the direction of gravity.

Prop. II.

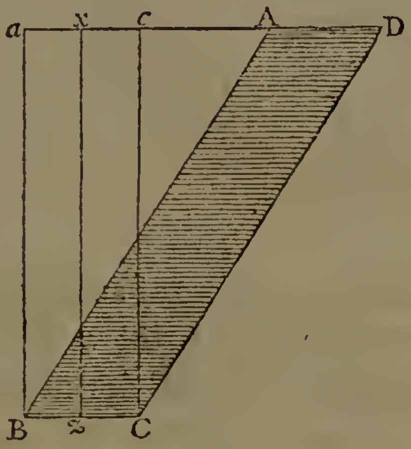
The preſſure upon any particle of a fluid, whoſe denſity is uniform, is in proportion to it's perpendicular diſtance from the ſurface of the fluid.

10. Caſe 1. Let $ABCD$ be a veſſel filled with ſuch a fluid, and draw xz perpendicular to the horizon. Now the preſſure is in proportion to the weight, or number of incumbent particles; and as the denſity of the fluid is uniform, and conſequently all the particles are at the ſame diſtance from each other, we have xy : xz :: the number of particles incumbent upon y : the number incumbent upon z :: the preſſure on y : the preſſure on z.



Caſe 2. Hence if the ſides of the veſſel $ABCD$ be not perpendicular to the bottom, the preſſure upon

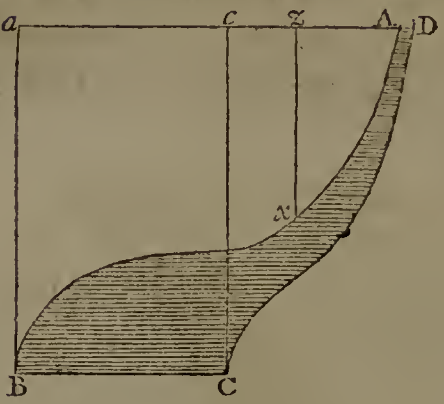
any point of BC will ſtill be in proportion to it's perpendicular depth. For produce DA to a, and draw Ba, Cc pendicular to Da; and conceive $DaBC$ to be a veſſel filled with the ſame fluid; then the preſſure



on z is as zx, there being actually that perpendicular depth of the fluid incumbent upon it. Now inſtead of ſuppoſing the fluid $ABCD$ to be kept in it's poſition by the other part of the fluid AaB, conceive it to be kept in that poſition by the ſide AB of the veſſel, and the effect muſt remain the ſame; for it can manifeſtly make no difference, by what body the fluid $ABCD$ is kept in it's place. Hence, whatever be the form of the ſides, the preſſure on BC will be the ſame as if the ſides were perpendicular to the ſurface of the fluid, and the perpendicular height the ſame.

11. That the preſſure at any point x is equivalent to the weight of a column xz, appears from this experiment, that if any where in AB, an hole be made perpendicular to Aa, a perfect fluid will ſpout upwards very nearly as far as Aa, the defect being no more than what may be ſuppoſed to ariſe from the reſiſtance

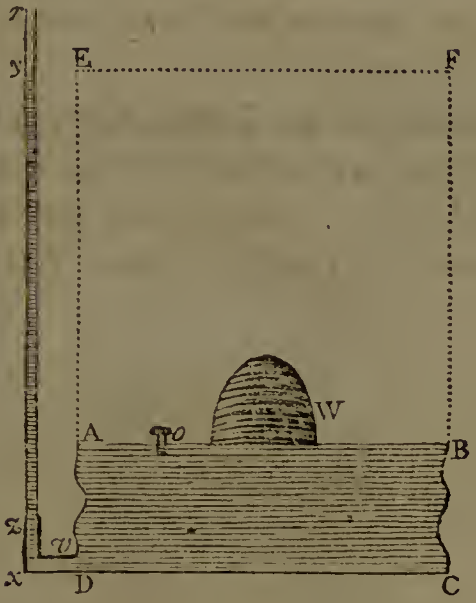
of the air, and the falling back of the upper parts of



the fluid. The want therefore of an actually incumbent column xz of fluid at any point x of AB is ſupplied by the reaction of the ſide AB downwards againſt the fluid. Alſo, let any two tubes be connected, and however they may differ in magnitude or inclination, if a fluid be put into one, it will always riſe to the ſame perpendicular altitude in the other. Theſe experiments therefore alſo prove, that fluids preſs in proportion to their perpendicular depths.

12. Upon theſe two principles, that fluids preſs equally in all directions, and in proportion to their perpendicular depths, depends the explanation of the following experiment, called the *hydroſtatical Paradox*. AB, CD are two round parallel boards, connected by leather like a pair of bellows; zxv is a braſs pipe entering into the cavity, and into which the glaſs tube zr is fixed. Through an orifice at *o* a quantity of water is poured in, to keep the top and bottom at ſome diſtance from each other, and then it is ſtopped by a ſcrew. Now let a very large weight W be laid upon the top AB, and the water in the tube will riſe to ſome height xy and there it will reſt, the

ſmall weight of water in *xy* balancing the weight *W*,

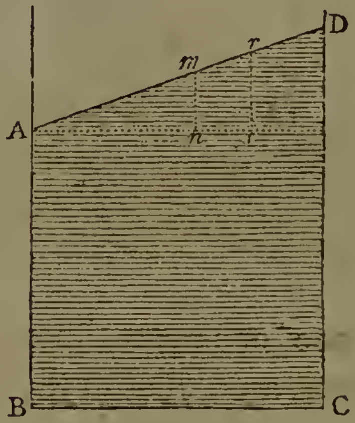


which may be 1000 or 10000 times greater than the weight of that water, according to the ſize of the tube; and this weight *W* will always be found equal to the weight of a cylinder *ABFE* of fluid, whoſe ſurface *EF* is upon a level with *y*. The principles before mentioned will thus explain this extraordinary circumſtance. The fluid at *x*, the bottom of the tube, is preſſed with a force proportional to the perpendicular altitude *xy*; this preſſure is communicated horizontally in the direction *xDC* to all the particles, and the preſſure ſo communicated acts equally in all directions; the preſſure therefore downwards upon the bottom *DC* is juſt the ſame as it would be, if *DEFC* were a cylinder of water; and it is manifeſt that if we take away the part *ABFE*, and place an equal weight *W* to act upon the other part *ABCD*, the effect will remain the ſame, or there will be an equilibrium.

PROP. III.

The ſurface of every fluid at reſt is horizontal.

13. For conceive the ſurface *AD* not to be horizontal, and draw *Ans* parallel to the horizon, and *mn*, *rs* perpendicular to it. Then (Art. 10) the preſſures at *s* and *n* are as *rs* and *mn*; and (Art. 8) fluids preſſing equally in all directions, the particle at *n* would be driven towards *A*; the fluid therefore would not remain at reſt. But when *AB* becomes horizontal, theſe preſſures become equal, and the fluid remains at reſt.

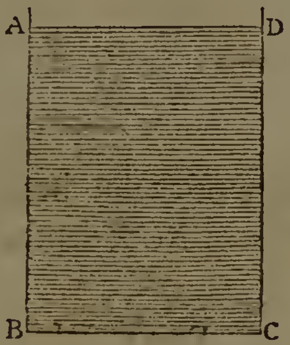


14. Cor. In like manner it appears, that if there be two fluids in the ſame veſſel which do not mix, their common ſurface will be parallel to the horizon.

PROP. IV.

If the ſides of a veſſel ABCD, filled with a fluid, be perpendicular to it's bottom which is parallel to the horizon, the preſſure upon the bottom will be equal to the weight of the fluid.

15. For the reaction of the ſides of the veſſel againſt the fluid, being perpendicular to the ſides, or parallel to the bottom, it can neither increaſe nor diminiſh the preſſure of the fluid againſt the bottom; alſo the force of gravity acting parallel to the ſides, they can take off no part of the weight of the fluid, nor increaſe it's preſſure upon BC; the preſſure therefore upon the bottom muſt be ſimply the weight of the fluid.



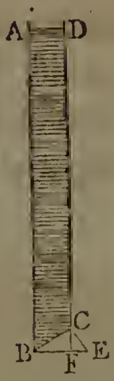
16. Cor. Hence, (Art. 36) in different veſſels of this deſcription, containing different fluids, the preſſures are as the areas of the bottoms × depths × ſpecific gravities, becauſe the magnitude = the area of the bottom × depth.

Prop. V.

Let the bottom BC be oblique to the ſides, and alſo ſo ſmall, that the whole may be conceived to be of the ſame depth; then the preſſure perpendicular to BC will be as BC × depth.

17. For the number of particles in contact with BC is as BC. Alſo, fluids preſs equally in all directions, and in proportion to their depths (Art. 8. 10), and the

whole preſſure on BC muſt be as the number of particles × the preſſure of each; hence the preſſure perpendicular to BC is as $BC\times$ depth.



Prop. VI.

The preſſure exerted upon BC downwards, *or in the direction of gravity, is equal to the weight of the fluid.*

18. Draw CE perpendicular to BC, BE to BA, and CF to BE. Now let CE repreſent the preſſure (P) perpendicular to BC, which reſolve into CF, FE, then CF is that part which acts downwards; alſo let p repreſent the preſſure downwards, and π the preſſure which would act upon BF as the bottom, or the weight of the fluid by Art. 15. Hence,

$$\begin{array}{l} p : P :: CF : CE :: \text{(by ſim. trian.) } BF : BC, \\ P : \pi :: BC\times\text{depth} : BF\times\text{depth} :: BC : BF. \\ \hline \therefore p : \pi :: BF\times BC : BC\times BF; \end{array}$$

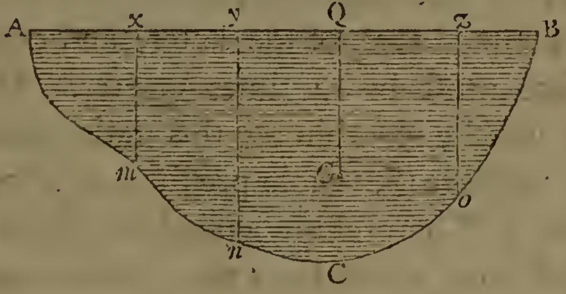
hence $p=\pi$.

The ſame may be inferred directly, from the reaſoning in Art. 15.

PROP. VII.

The preſſure of a fluid againſt any ſurface, in a direction perpendicular to it, varies as the area of the ſurface multiplied into the depth of it's center of gravity below the ſurface of the fluid.

19. Let ABC be a veſſel, AB the ſurface of the fluid; divide ACB into an indefinite number of parts m, n, o, &c. let G be the center of gravity of the ſurface ACB, and draw GQ, mx, ny, oz, &c. perpendicular to AB. Now every part of the indefinitely ſmall ſurface m may be conceived to be at the ſame perpendicular depth mx; alſo, (Art. 10) the preſſure at m is in proportion to it's depth, and that preſſure is exerted equally in all directions; hence, the preſſure on m perpendicular to that ſurface is as $m\times mx$; conſequently the whole preſſure exerted on ACB, in a direction perpendicular to the ſurface at every point, will be as $m\times mx+n\times ny+o\times\ z+$&c. But (by Mechanics, Art. 172.) if we conſider m, n, o, &c. as repreſenting weights in proportion to their reſpective magnitudes, and the whole area ACB to repreſent a proportional weight, then $m\times mx+n\times ny+o\times oz$ +&c.



$+\&c. = ACB \times GQ$; hence the whole preſſure perpendicular to the ſurface varies as $ACB \times GQ$.

20. Cor. 1. The ſame preſſure is equal to the weight of a cylinder of that fluid, the area of whoſe bottom is equal to the ſurface ACB, and altitude GQ. For the areas preſſed, and the depths of the centers of gravity, are equal in the two caſes, therefore the preſſures are equal. But (Art. 15) the preſſure on the bottom of a cylinder is equal to the weight of the fluid contained in it; conſequently the preſſure on the bottom ACB is equal to the ſame weight.

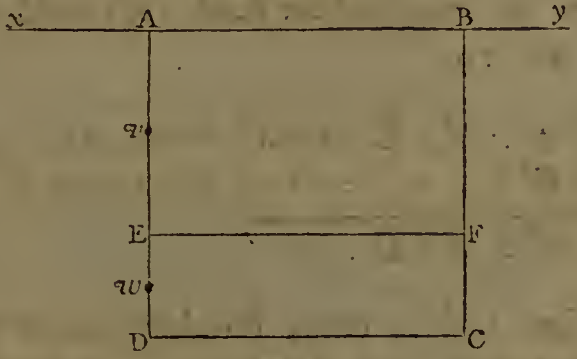
21. Cor. 2. The preſſures of different fluids againſt different ſurfaces, will be as the areas × depths of their centers of gravity × ſpecific gravities of the fluids. For it is manifeſt that, cæteris paribus, the preſſures muſt be as the weights, or as the ſpecific gravities (Art. 3); thus, if the ſame veſſel be filled with mercury and water, the ſpecific gravity, or weight, of the former will be 14 times that of the latter, and conſequently the preſſure muſt be 14 times greater. Hence, for different veſſels, combining this ratio with that in Art. 19. we have the whole preſſures as above.

22. Cor. 3. The preſſure againſt the ſide of a cubical veſſel filled with a fluid $= \frac{1}{2}$ the preſſure againſt the bottom; for the area preſſed is the ſame, and the depth of the center of gravity in the former caſe $= \frac{1}{2}$ that in the latter. Hence the preſſure againſt the ſide $= \frac{1}{2}$ the weight of the fluid.

23. Cor. 4. If a cylinder, the altitude of which is a, and diameter of it's baſe d, be filled with a fluid; then if $p = 3{,}14159$ &c. the area of the baſe $= \frac{1}{4}pd^2$, and the area of the ſides $= pda$; hence the preſſure

on the bottom : the preſſure on the ſide :: $\frac{1}{4}pd^2a$: $\frac{1}{2}pda^2$ :: $d : 2a$. If two equal cylinders be filled, one with mercury and the other with water, then, the preſſure on the baſe of the former : the preſſure on the ſides of the latter :: $14d : 2a :: 7d : a$.

24. Cor. 5. Let xy be the ſurface of a fluid, $ABCD$



a ▭ perpendicular to it; draw EF parallel to AB, and biſect AE in v, ED in w; then Av, Aw will be the depths of the centers of gravities of $ABFE$ and $EFCD$. Hence the preſſures on theſe ▭s are as $ABFE \times Av : EFCD \times Aw :: AE \times Av : ED \times Aw :: AE \times \frac{1}{2}AE : \overline{AD - AE} \times \frac{AD + AE}{2} :: AE^2 : AD^2 - AE^2$.

Prop. VIII.

If a veſſel be filled with a fluid, the preſſure on any part : the whole weight of the fluid :: the area of that part × the depth of it's center of gravity : the ſolid content of the fluid.

24. By Art. 15. the preſſure on the baſe of a cylinder filled with fluid = it's weight; alſo, the weight of

the ſame fluid is in proportion to it's ſolid content. Let a veſſel of any form be filled with a fluid, and let $A=$ any part of it's ſurface, $D=$ the depth of it's center of gravity, $P=$ the preſſure upon it, and $W=$ the whole weight of the fluid, $S=$ it's ſolid content; and let a cylinder, whoſe baſe $=a$ and altitude d, be filled with the ſame fluid, and let p be the preſſure on it's bottom, w the weight of the fluid, s it's ſolid content; then by Art. 19.

$$\left.\begin{array}{rl} & P : p :: A\times D : a\times d \\ \text{Alſo,} & w : W :: \quad s \quad : \quad S \end{array}\right\} \begin{array}{l}\text{but by Art. 15. } w=p; \\ \text{alſo } s=a\times d; \text{ hence}\end{array}$$

$$\overline{P : W :: A\times D : \; S.}$$

25. Cor. 1. If a cone, ſtanding on it's baſe, be filled with a fluid, and $A=$ the baſe, we have $S=\frac{1}{3}A\times D$; conſequently the preſſure on the baſe $=$ three times the weight of the fluid.

26. Cor. 2. If a hollow ſphere be filled with a fluid, the preſſure P againſt the whole internal ſurface $A=$ three times the weight of the fluid; for the center of gravity of the ſurface is in the center of the ſphere, whoſe depth D below the upper point of the fluid muſt therefore be equal to the radius R of the ſphere, and $S=\frac{1}{3}A\times R=\frac{1}{3}A\times D$.

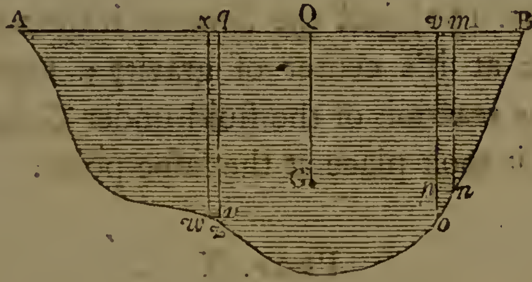
27. Hitherto we have conſidered the whole preſſure of a fluid againſt any ſurface in a direction *perpendicular* to every point of it; but in this caſe, the effect on one part may partly deſtroy the effect on another, by their not acting in the ſame direction. Since we do not therefore thus get the whole joint effect in any direction, let us next conſider, what is the whole preſſure againſt any plane, in the direction of gravity.

PROP.

Prop. IX.

The preſſure of a fluid downwards *againſt the ſides and bottom of any veſſel, is equal to the weight of the whole fluid, provided there be, over every part of the ſides and bottom, a perpendicular column of the fluid reaching to the ſurface*

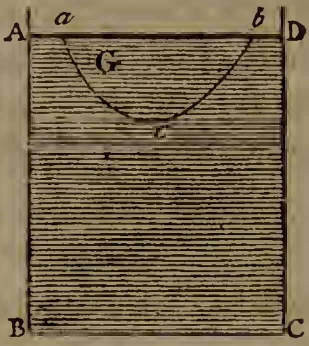
28. Let *AwzonB* be ſuch a veſſel filled with a fluid whoſe ſurface is *AB*, perpendicular to which, conceive *vmno*, *qxwz*, &c. to be indefinitely ſmall priſmatic columns into which the whole is divided; then (Art. 17.) the preſſure downwards of every column is equal to the weight of the column of fluid; hence the whole preſſure downwards upon the ſurface *ACB* is always equal to the whole weight of the fluid.



29. Cor. 1. As the preſſure of every column downwards is equal to it's gravity, the joint effect of all the columns, or of the whole fluid, is the ſame as the gravity of the whole if it had been ſolid, and conſequently (Mechanics, Art. 160) the effect is the ſame as if all the power was concentrated in the center of gravity.

30. Cor. 2. Hence if *ABCD* be a veſſel of fluid,

and *acb* be any part *G* of the fluid, it's action downwards, muſt be equal to the reaction of the fluid under it upwards; but the effect of *G* downwards, is

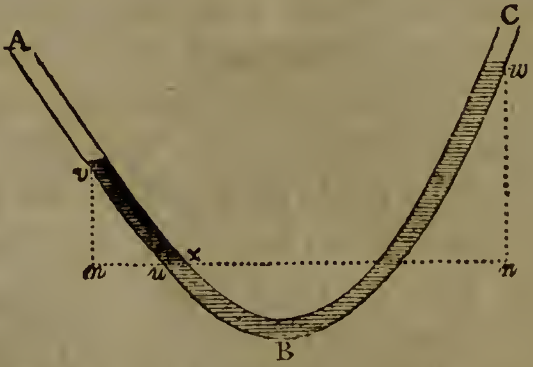


(Mech. Art. 160) juſt the ſame as if the whole effect took place at it's center of gravity; therefore the effect of the reaction of the fluid under *G* muſt be the ſame as if it took place at the ſame point.

PROP. X.

If two fluids communicate in a bent tube, their perpendicular altitudes above the plane, where they meet, are inverſely as their ſpecific gravities.

31. Let *ABC* be the tube, *ux* the plane where

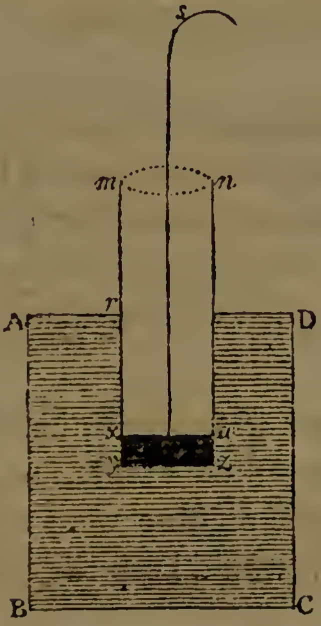


the two fluids meet, ſtanding at *v* and *w*; draw

$muxn$ parallel to the horizon, and vm, wn perpendicular to it. Let S, s repreſent the ſpecific gravities of the two fluids in vu, uw; then the area of the ſection ux being common to both fluids, the preſſure of each fluid at that ſection will (Art. 21) be as it's perpendicular depth × it's ſpecific gravity; but as the fluids are at reſt, their preſſures muſt be equal; hence $S \times vm = s \times wn$, therefore $S : s :: wn : vm$.

32. Cor. 1. Hence the ſame fluid will ſtand at the ſame altitude on each ſide; for if $S = s$, then $wn = vm$. If therefore a pipe convey a fluid from a reſervoir, it can never carry it to a place higher than the ſurface of the fluid in the reſervoir.

33. Cor. 2. If $ABCD$ be a veſſel of fluid, $mxwn$ a hollow cylinder, to whoſe bottom a cylindrical body $wxyz$, of greater ſpecific gravity than the fluid, may be ſo cloſely fitted, that the fluid cannot enter; then if



if this body be kept in that poſition by the ſtring s, and the whole be immerſed perpendicularly in the veſſel, until yr be to yx as the ſpecific gravity of the body to that of the fluid, the body will remain ſuſpended without the aſſiſtance of the ſtring. For, we may conſider the body $wxyz$ juſt the ſame as if it were a fluid of the ſame ſpecific gravity, and conſequently it will reſt when the altitudes of the body and fluid above yz are inverſely as their ſpecific gravities.

Prop. XI.

The aſcent of a body in a fluid of greater ſpecific gravity than itſelf, ariſes from the preſſure of the fluid upwards againſt the under ſurface of the body.

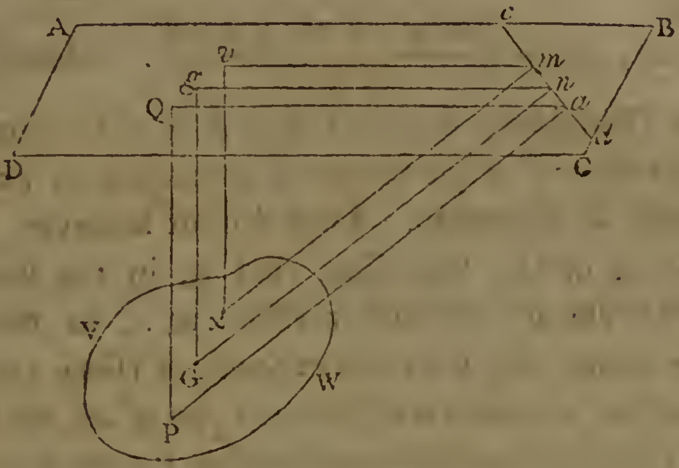
34. For if the body be placed upon the bottom of the veſſel in which the fluid is, and be ſo cloſely fitted to it, that no part of the fluid can get under it, it will remain at reſt; but if it be lifted up, ſo that the fluid can get under it, it will immediately riſe.

Def. The *center of preſſure* is that point of a ſurface, againſt which any fluid preſſes, to which if a force equal to the whole preſſure were applied, in a contrary direction, it would keep the ſurface at reſt.

Prop. XII.

To find the center of preſſure of a plane ſurface.

35. Let *ABCD* be the ſurface of the fluid, *VW* the plane, which being produced, let *cd* be it's



interſection with the ſurface, *P* the center of preſſure, and *G* the center of gravity; and conceive the whole plane to be divided into an indefinite number of indefinitely ſmall parts, of which one is x; draw PQ, Gg, xv perpendicular to the ſurface, and Pa, Gn, xm perpendicular to cd; and join Qa, gn, vm; then it is manifeſt, that the triangles PQa, Ggn, xvm are ſimilar. Now the preſſure on x perpendicular to *VW* is (Art. 17.) as $x \times xv$; and (Mechanics, Art. 92) it's effect to turn the plane about cd is as $x \times xv \times xm$; but (ſim. trian.) $Gn : Gg :: xm : xv = xm \times \frac{Gg}{Gn}$; hence the effect of the preſſure at x to turn the plane about cd is as $x \times xm^2 \times \frac{Gg}{Gn}$; therefore the whole effect is as the *ſum of all the* $x \times xm^2 \times \frac{Gg}{Gn}$. But if $A =$ the area of *VW*, the preſſure on $VW = A \times Gg$; therefore the effect of that preſſure at *P*, to turn the plane about cd, is as $A \times Gn \times Pa$. Hence $A \times Gg \times Pa =$ *ſum of all the* $x \times xm^2 \times \frac{Gg}{Gn}$, conſequently, $Pa = \frac{\text{ſum of all the } x \times xm^2}{A \times Gn}$. Hence it appears (Fluxions, Art. 62) that P is at the ſame diſtance from cd as the center of percuſſion is, cd being the axis of ſuſpenſion. They do not however, in general, lie in the ſame line, that is, in the line nG; for the efficacy of the preſſure at x, to turn the plane about nG, is as $x \times xv \times mn$, or (ſince xv varies as xm) as $x \times xm \times mn$; but the *ſum of all the* $x \times xm \times mn$ is not generally $= 0$, therefore the whole preſſure will not balance itſelf upon the line Gn. The ſituation of the line Pa muſt be determined, by making the *ſum of all the* $x \times xm \times mn = 0$, which, in any particular caſe, may be determined by a fluxional calculation. It is not therefore true, in general, that the centers of preſſure and percuſſion are the ſame point, as is aſſerted by writers on this ſubject; indeed there are but very few caſes in which they coincide.

* * *

SECTION

SECTION II.

ON THE SPECIFIC GRAVITIES OF BODIES.

PROP. XIII.

THE weight of a body varies as it's magnitude and ſpecific gravity conjointly.

36. For it is manifeſt, that if you vary the magnitude of any body in the ratio of $M : m$, continuing it's ſpecific gravity the ſame, you will alter the quantity of matter, and conſequently the weight, in the ſame ratio. If you alter the ſpecific gravity of the body in the ratio of $S : s$, continuing it's magnitude the ſame, you will alter the weight in the ſame ratio, (Art. 3). Hence, by the compoſition of ratios, if you alter both the magnitude and ſpecific gravity, you will alter the weight in the ratio of $M \times S : m \times s$.

As the weight of a body is in proportion to it's quantity of matter, the quantity of matter is as the magnitude and ſpecific gravity conjointly, or as the magnitude and denſity conjointly, the ſpecific gravity and denſity varying in the ſame ratio, (Art. 5).

37. A cubic foot of rain water weighs 1000 ounces avoirdupoiſe, the ſpecific gravity of which call s; let W be the weight of another body whoſe magnitude in cubic feet is M, and ſpecific gravity S. Hence, $W : 1000 :: M \times S : 1$ foot $\times s$; if therefore we aſſume $s = 1000$ as a ſtandard with which we may compare the ſpecific gravities of other bodies, we have $W = M \times S$. To reduce it to the meaſure in cubic inches, correſponding to the ſpecific gravity of water repreſented by unity, we have, $1728 : 1 :: 1000 : ,5787$ the weight of a cubic inch of rain water; hence, $W : ,5787 :: M \times S : 1$ inch $\times 1 (s)$, therefore, $W = 5787$, $M \times S$, M being the magnitude in cubic inches, and the ſpecific gravity of water unity. Now a troy ounce : avoirdupoiſe ounce :: 480 : 437,5, for an avoirdupoiſe ounce contains 437,5 grains troy. Hence, to reduce W to troy weight, as the troy ounce is the greater, the weight expreſſed in troy ounces will be leſs in the ſame proportion; therefore, $480 : 437,5 :: ,5787\ M \times S : ,52746\ M \times S = W$ the weight in troy ounces, $= 253,18\ M \times S$ grains. If $M = 1$, $S = 1$, this gives 253,18 grains for the weight of a cubic inch of rain water *.

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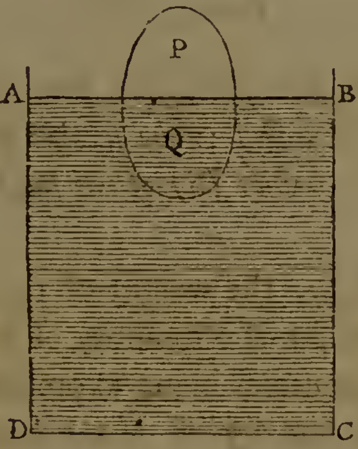
* Hence, we may very accurately determine the diameter d of a ſphere whoſe ſpecific gravity is s, that of water being unity. For the content of a ſphere whoſe diameter $= 1$ is 0,5236; therefore, $1 : 0,5236 :: 253,18$ grains (the weight of 1 cubic inch) : 132,428 grains, the weight of a ſphere of water whoſe diameter is 1 inch. Hence, ſince the weights are as the magnitudes and ſpecific gravities conjointly, and the magnitudes of ſpheres are as the cubes of their diameters, we have $132,428\ sd^3 = w$, conſequently, $d = \left(\frac{w}{s}\right)^{\frac{1}{3}} \times ,19612$.

Prop. XIV.

If a body float on a fluid, it diſplaces as much of the fluid as is equal to the weight of the body.

38. For the body $P+Q$ is ſupported by the preſſure of the fluid upwards againſt the part immerſed; alſo, the ſame preſſure ſupported a quantity of fluid equal in bulk to Q, before the body was immerſed, that ſpace being occupied by the fluid; and as the ſame preſſure muſt ſuſtain the ſame weight, when there is an equilibrium, the weight of the body muſt be equal to the weight of a quantity of fluid equal in bulk to Q.



Prop. XV.

If a body float on a fluid, the centers of gravity of the body and of the fluid diſplaced, muſt, when the body is at reſt, be in the ſame vertical line.

39. For (Art. 30) the effect of the preſſure of the fluid upwards is the ſame as if the whole took place at the center of gravity of the fluid diſplaced; if therefore this action of the fluid upwards againſt the body in a vertical line do not paſs through the center of gravity of the body, it muſt (Mech. Art. 164) give the body a rotatory motion.

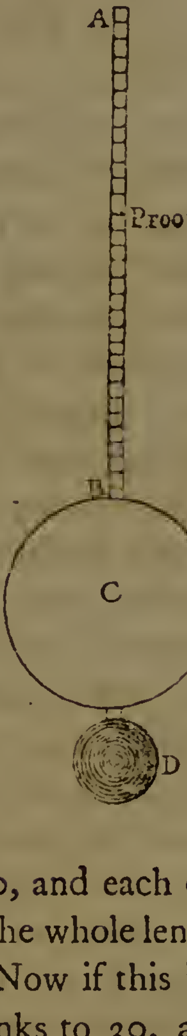
Prop. XVI.

If a body float on a fluid, the part (Q) *immerſed* : *the whole body* ($P+Q$) :: *the ſpecific gravity* (s) *of the body* : *the ſpecific gravity* (S) *of the fluid.*

40. By Art. 37. the weight of the body is $\overline{P+Q}\times s$, and the weight of a quantity of fluid equal to Q is $Q\times S$, the magnitudes being expreſſed in cubic feet, and the ſpecific gravity of water being 1000; but (Art. 39) their weights are equal; hence, $\overline{P+Q}\times s = Q\times S$; conſequently, $Q : P+Q :: s : S$. We here ſuppoſe the part P to be in vacuo. But when bodies float upon a fluid, the part P being in the air, this rule will not be accurately true; near enough ſo, however, for all practical purpoſes. The effect of the air, if neceſſary, may be computed from Art. 52.

41. Cor. Hence, if the ſame body float upon two different fluids, the parts immerſed will be inverſely as the ſpecific gravities of the fluids; for in this caſe $P+Q$ and s being conſtant, Q varies inverſely as S. Upon this principle is conſtructed the HYDROMETER, an inſtrument for meaſuring the ſpecific gravities of ſuch fluids, as do not differ much in their ſpecific

gravities. *AB* is a graduated ſtem fixed to a hollow globe *C*, which is annexed to another ſphere *D*, into which mercury or ſhot is put, in order to make the inſtrument ſink in the fluid, and keep it vertical. Now let us ſuppoſe the whole bulk of the inſtrument to be



repreſented by 4000, and each of the diviſions by 1 of ſuch parts, and let the whole length of the ſtem contain 50 of ſuch parts. Now if this inſtrument be put into one fluid, and it ſinks to 30, and in another, it ſinks to 20, then the parts immerſed will be 3970 and 3980 reſpectively; and the ſpecific gravities of theſe two fluids, being inverſely as the parts immerſed, will be as 3980 : 3970. This can only be applied to thoſe fluids which come within the extent of the ſcale. This inſtrument therefore generally conſiſts of two ſtems, one of which can be taken off and the other put on, the ſtems being adapted to fluids of different ſpecific gravities, one meaſuring what the other will not.

Mr. Nicholson has lately made a conſiderable improvement in this inſtrument, by placing a ſmall braſs cup on the top of the ſtem, into which ſmall weights may be put, ſo as to ſink it into different fluids to the *ſame* point of the ſtem. In this caſe, the part immerſed being the ſame, the ſpecific gravities of the fluids will be as the whole weights, which are known, from knowing the weight of the inſtrument, and the weights added.

This inſtrument is alſo uſed to find whether a fluid is above or below proof. Proof ſpirits conſiſt, half of pure ſpirits, called alcohol, and half of water; the inſtrument put into this mixture, ſinks to *Proof* upon the ſcale. Now as water is heavier than the ſpirits, if there be more water than ſpirits, the ſpecific gravity will be greater than that of proof ſpirits, and conſequently the hydrometer will not ſink ſo far as proof, or the ſurface of the liquor ſtands below proof, and the ſtrength of the ſpirits is *below* proof: but if there be leſs water than ſpirits, the ſpecific gravity will be leſs than that of proof ſpirits, and therefore the hydrometer will ſink below proof, or the ſurface of the fluid will ſtand above proof, and the ſpirits are *above* proof. This is the inſtrument which the Officers of Exciſe generally uſe when they examine liquors, in order to determine their ſtrength, for the purpoſe of aſcertaining the duty.

Prop. XVII.

The weight which a body loſes when wholly immerſed in a fluid, is equal to the weight of an equal bulk of the fluid.

42. Firſt, let the body be of the *ſame* ſpecific gravity as the fluid, and then it is manifeſt that it will remain at reſt, becauſe (Art. 3) it is of the ſame weight as an equal bulk of the fluid, and therefore the preſſure upwards of the fluid beneath will ſupport the body equally as it ſupported the fluid which occupied the ſpace before the body was put in; in this caſe, the body is ſaid to have loſt all it's weight, or the weight of an equal bulk of fluid. Now let the ſpecific gravity, and conſequently the weight of the body, be increaſed, then the preſſure of the fluid upwards againſt the body ſtill continuing the ſame, that action muſt ſtill take off the ſame weight from the body, that is, the weight of an equal bulk of fluid.

When we ſay, that a body loſes part of it's weight in a fluid, we do not mean that it's abſolute weight is leſs than it was before, but that it is partly ſupported by the reaction of the fluid under it, ſo that it requires a leſs power to ſupport it.

43. Cor 1. Hence, when a body is weighed in air, in order to get it's abſolute weight, we muſt add to it the weight of an equal bulk of air. If, for example, a body, whoſe magnitude is one cubic foot, weigh 1500 ounces troy in air, we muſt add 21 pennyweights to it, which is the weight of a cubic foot of air, and it gives 1500 oz. 21 dwts. the real weight of the body, or it's weight in vacuo.

44. Cor. 2. Hence alſo it follows, that if two bodies of the ſame weight in air, be put into a denſer fluid, the ſmaller body will preponderate, ſince it loſes a leſs weight than the other. And if they weigh equally in any fluid, and then be brought into a rarer medium, the greater will preponderate, becauſe having loſt more weight in the denſer fluid than the other body, when carried into a rarer fluid it will regain more weight, and therefore will weigh moſt in the lighter fluid.

Prop. XVIII.

A body immerſed in a fluid, aſcends or deſcends with a force equal to the difference between it's own weight and the weight of an equal bulk of fluid, the reſiſtance of the fluid not being conſidered.

45. Let W be the weight of the body, w the weight of an equal bulk of the fluid. Now we may conſider the body as deſcending by it's own weight W, and (Art. 42) as oppoſed in it's deſcent by w; hence, when W is greater than w the body deſcends, and when W is leſs than w it muſt aſcend, and, in both caſes, by the difference between W and w, as they oppoſe each other.

46. Cor. Let the ſpecific gravity of the fluid : that of the body :: 1 : b; then, (Art. 3) $w : W :: 1 : b$; hence $w = \frac{W}{b}$, conſequently the relative gravity, or weight of the body in the fluid, $= W - \frac{W}{b}$.

Prop.

PROP. XIX.

The weight (w) which a body loſes when immerſed in a fluid : it's whole weight (W) :: the ſpecific gravity of the fluid (s) : the ſpecific gravity of the body (S).

46. For (Art. 43) w is the weight of a bulk of fluid equal to the bulk of the body, the weight of which is W; but (Art. 3) the weights of equal bulks are as their ſpecific gravities; conſequently $w : W :: s : S$.

Ex. If a body weigh 12 lb. in air, and 7 lb. in water, then 5 lb. is the weight loſt; hence, 5 : 12 :: the ſpecific gravity of water : the ſpecific gravity of the body. Thus we compare the ſpecific gravities of bodies and fluids of leſs ſpecific gravities.

47. Cor. 1. Hence, $S = \frac{s \times W}{w}$; if therefore s be given, that is, if different bodies be weighed in the *ſame* fluid, then will S vary as $\frac{W}{w}$.

Ex. If a body A weigh 9 lb. in air, and 7 lb. in any fluid, and another body B weigh 13 lb. in air, and 6 lb. in the ſame fluid, then the ſpecific gravity of A : that of B :: $\frac{9}{2} : \frac{13}{7}$ :: 63 : 26. Thus we compare the ſpecific gravities of two bodies.

48. Cor. 2. Hence alſo, $s = \frac{S \times w}{W}$; if therefore S and W be given, that is, if the *ſame* body be weighed in different fluids, then will s vary as w.

Ex. If a body loſe 6 lb. in one fluid, and 5 in another, the ſpecific gravities of theſe fluids are as 6 : 5. Thus we compare the ſpecific gravities of two fluids.

If one of the fluids be mercury, the body muſt be either gold or platina, theſe being the only two metals which will ſink in mercury. It is better, however, in this caſe, to compare the mercury with water (Art. 46), weighing the mercury in the water by putting it into a ſmall glaſs veſſel ſuſpended from the ſcale, firſt balancing the veſſel in the water.

49. The effect of the air in diminiſhing the weight of a body is not here conſidered. If the body which is weighed in the fluid be wood, it ſhould firſt be well rubbed over with greaſe, or varniſhed, to prevent it from imbibing any of the fluid.

50. Since the ſpecific gravities of fluids vary when their temperatures vary, in comparing the ſpecific gravities of different fluids, we muſt firſt reduce them to ſome one temperature, as a ſtandard. This temperature is arbitrary; and it muſt be obſerved that, from the *different* expanſions of fluids for the *ſame* variation of temperature, the proportion of the ſpecific gravities at different degrees of temperature will be different. Many ſolid bodies are alſo ſubject to a variation of their ſpecific gravities, from the variation of their temperatures.

Prop. XX.

To find the ſpecific gravity of a body Q, which is lighter than the fluid in which it is weighed.

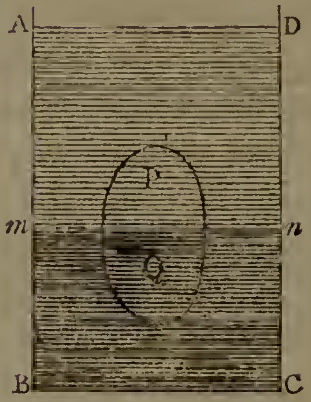
51. Connect Q with another body H which is heavier than the fluid, ſo that together they may ſink; let the weight of P in the fluid be a, the weight of $P+Q$ in the fluid be b, the weight of Q out of the fluid be d, and the weight of a bulk of fluid equal to Q be e. Now, as Q is of leſs ſpecific gravity than the fluid, it will (Art. 45) aſcend with the force $e-d$. Alſo, as Q by itſelf would aſcend, when it is connected with P, they will together have a leſs deſcending power in the fluid than P of itſelf would have, by the power of Q's aſcent; now the weight or deſcending power of P, in the fluid is a, and the weight or deſcending power of $P+Q$ in the fluid is b, therefore the difference $a-b$ gives the aſcending power of Q; hence, $e-d=a-b$, and $e=a-b+d$; but (Art. 3) the weights of equal bulks are as the ſpecific gravities; hence, the ſpecific gravity of Q : the ſpecific gravity of the fluid $:: d : a-b+d$.

PROP. XXI.

If a lighter fluid reſt upon a heavier, and their ſpecific gravities be as $a : b$, *and a body, whoſe ſpecific gravity is* c, *reſt with one part* P *in the upper fluid, and the other part* Q *in the lower, then* $P : Q :: b-c : c-a$.

52. The body will reſt when it has diſplaced as much of the two fluids as is equal in weight to itſelf, for the reaſon given in Art. 38. Now (Art. 37) the weight of the body is $c\times\overline{P+Q}$; alſo, the weight of the lower fluid diſplaced is $b\times Q$, and of the upper fluid, $a\times P$; hence, $a\times P+b\times Q=c\times\overline{P+Q}$

$c\times\overline{P+Q}=c\times P+c\times Q$; therefore, $b\times Q-c\times Q=$



$c\times P-a\times P$, or $\overline{b-c}\times Q=\overline{c-a}\times P$; hence, $P:Q$ $::b-c:c-a$.

53. Cor. Hence, $Q:P+Q::c-a:b-a$, and if a be ſo ſmall that it may be neglected, then $Q:$ $P+Q::c:b$, as in Art. 40.

Prop. XXII.

If a and b be the ſpecific gravities of two fluids to be mixed together, P and Q their magnitudes, and c the ſpecific gravity of the compound, then $P:Q::b-c:$ $c-a$.

54. By Art. 37. the weight of P is $a\times P$, the weight of Q is $b\times Q$, and the weight of the compound is $c\times\overline{P+Q}$; but the weight of the compound muſt be equal to the ſum of the weights of the two parts; hence (as in Art. 52) $P:Q::b-c:c-a$.

55. It is here ſuppoſed, that the magnitude $P+Q$, of the compound, is equal to the ſum of the magnitudes of the two parts when ſeparate. But it very often happens,

pens, that the magnitude of the mixture is leſs than this ſum, owing, probably, partly to the conſtituent particles of the different fluids being of different magnitudes, and partly to their chemical affinity. This is called a *penetration of dimenſions.* Thus, for inſtance, if a pint of water and a pint of oil of vitriol be mixed together, the mixture will not make a quart. The ſpecific gravity of the compound is manifeſtly increaſed by this circumſtance.

Ex. Let the ſpecific gravity of gold be 19, of ſilver 11, and of the compound 14; then the magnitude of the ſilver in the mixture : the magnitude of the gold :: 19 − 14 : 14 − 11 :: 5 : 3.

A certain quantity of gold having been given by King HIERO to make him a crown, the artiſts ſecreted part of the gold, and ſubſtituted the ſame weight of ſilver. This being ſuſpected, ARCHIMEDES was employed to diſcover the cheat; but it is not related in what manner he did it, except that by going into a bath, the riſing of the water ſuggeſted to him the method of finding the magnitudes of irregular bodies.

56. If we want to find the proportion of the *weights* of each body, we muſt take the ratios of their magnitudes, and of their reſpective ſpecific gravities conjointly; hence, the weights of P and Q are as $a \times \overline{b-c} : b \times \overline{c-a}$. In the above example, therefore, the weight of the ſilver : the weight of the gold :: $11 \times 5 : 19 \times 3$ :: 55 : 57.

TABLE OF SPECIFIC GRAVITIES.

Refined gold	19637
Engliſh guinea	17793
Mercury	14019
Lead	11325
Refined ſilver	11087
Standard ſilver	10535
Biſmuth	9700
Copper of Japan	9000
Copper of Sweden	8843
Hammered braſs	8349
Caſt braſs	8100
Turbeth mineral	8235
Cinnabar, factitious	8200
Cinnabar, natural	7300
Elaſtic ſteel	7820
Soft ſteel	7738
Iron	7645
Pure tin	7471
Glaſs of antimony	5280
A pſeudo topaz	4270
A diamond	3400
Chryſtal glaſs	3150
Iſland chryſtal	2720
Rock chryſtal	2658
Common glaſs	2620
Fine marble	2704

Stone of a mean gravity	2500
Selenitis	2252
Sal gemmæ	2143
Nitre	1900
Alabaſter	1875
Dry ivory	1825
Brimſtone	1800
Dantzic vitriol	1715
Allum	1714
Borax	1714
Calculus humanus	1700
Oil of vitriol	1700
Oil of tartar	1550
Bezoar	1500
Honey	1450
Gum arabic	1375
Spirit of nitre	1315
Aqua fortis	1300
Pitch	1150
Spirit of ſalt	1130
Craſſamen of the human blood	1126
Spirit of urine	1120
Human blood	1054
Amber	1030
Serum of human blood	1030
Milk	1030
Urine	1030
Dry box-wood	1030
Sea-water	1030
Common water	1000
Camphire	996
Bees wax	955
Linſeed oil	932

Dry oak	925
Oil olive	913
Spirit of turpentine	864
Rectified ſpirit of wine	866
Dry aſh	800
Dry maple	755
Dry elm	600
Dry fir	550
Cork	240
Air	$1\frac{1}{5}$

The ſpecific gravity of rain water being here repreſented by 1000, and a cubic foot of rain water weighing 1000 ounces avoirdupoiſe, the numbers againſt each ſubſtance repreſents the weight of a cubic foot thereof in avoirdupoiſe ounces. The ſpecific gravities are ſubject to a ſmall degree of variation, ariſing from the variation of temperature of the air.

The ſcales which are made uſe of to weigh bodies, in order to determine their ſpecific gravities, are called the *Hydroſtatic Balance.*

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SECTION

SECTION III.

ON THE RESISTANCE OF FLUIDS.

57. THE reſiſtance of a body moving in a fluid ariſes from the inertia, the tenacity, and friction of the fluid, admitting the particles to be in contact. The latter cauſe, granting it to exiſt, is probably very ſmall; and the ſecond is, in moſt fluids, inconſiderable when compared with the inertia. The reſiſtance therefore, which we ſhall here conſider, is that ariſing from the inertia of the fluid.

Prop. XXIII.

If a plane ſurface move in a fluid with a velocity V, in a direction perpendicular to it's plane, the reſiſtance, within certain limits of the velocity, varies as V^2.

58. For the reſiſtance muſt vary as the number of particles which ſtrike the plane in a given time, multi-

plied into the force of each againſt the plane. Now the number of particles which the plane ſtrikes in a given time muſt evidently be in proportion to V; alſo, the force of each particle is as V, and as action and reaction are equal and contrary, the reaction of every particle of the fluid againſt the plane muſt be as V; hence the reſiſtance varies as $V \times V$, or as V^2. This is found, by experiment, to be very nearly true, when the velocity is ſmall.

59. This proof ſuppoſes, that after the plane ſtrikes a particle, the action of that particle immediately ceaſes, and the particle itſelf to be, as it were, annihilated; but the particles, after they are ſtruck, muſt neceſſarily be made to diverge and act upon the particles behind, which makes ſome difference between this theory and experiment. Alſo, by increaſing the velocity of the body, the action of the fluid behind it, to impel it in the direction of it's motion, will be diminiſhed, and conſequently the retardation will, on this account, be increaſed. Mr. Robins found, from experiment, that when a bullet moves with the velocity of ſound, or with a greater velocity (in which caſe, a vacuum is left behind the body, and the preſſure forwards from behind then ceaſes), that the reſiſtance is always greater than this law gives it. When bodies deſcend in fluids, ſuch as water, the reſiſtance is very nearly as V^2, becauſe the body can never acquire a velocity beyond a certain limit. We will therefore, in the articles here given upon reſiſtances, ſuppoſe the reſiſtance to vary as V^2. This law of reſiſtance was eſtabliſhed by Sir I. Newton, from a variety of experiments; ſee the *Principia*, Vol. II. Prop. 31, Scholium; alſo Mr. Parkinson's *Hydroſtatics*, page 26.

Prop. XXIV.

When different planes move in directions perpendicular to their ſurfaces, in different fluids, and with different velocities, the reſiſtances will be as the ſquares of their velocities × *the denſities of the fluids* × *the areas of the planes.*

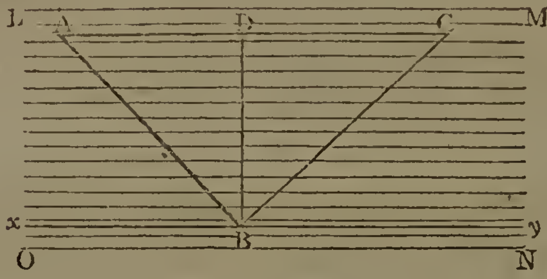
60. For by increaſing the denſity of the fluid, the number of particles ſtruck in the ſame time will be greater in the ſame proportion, and conſequently the reſiſtance will, cæteris paribus, be greater in the ſame ratio. Alſo, by increaſing the area of the plane, the greater will be the number of particles ſtruck in the ſame ratio, and therefore the reſiſtance will be greater in the ſame proportion. And (Art. 58) when the velocities vary, cæteris paribus, the reſiſtance varies as V^2. Hence, combining theſe ratios together, the reſiſtance will be as V^2 × the denſities of the fluids × areas of the planes.

Prop. XXV.

If a plane move obliquely in a reſiſting medium, with an uniform velocity, and after the reſolution of the force with which the plane ſtrikes the fluid, the whole of that part which acts perpendicular *to the plane takes effect, the reſiſtance perpendicular to the plane will vary as the ſquare of the ſine of the angle of inclination.*

61 Let AB be the plane moving in the medium $LMNO$ in the direction xy; draw AC parallel to xy, meeting BC perpendicular to AB in C; and let BD be

perpendicular to *AC*. Now the quantity of fluid which *AB* has to oppoſe by it's motion, being that which is contained between *AC* and *xy*, is manifeſtly in proportion to *BD*, or to the ſine of *BAC*, becauſe *AB* : *BD* :: rad. = 1 : ſin. *BAD* or *BAC*, and the firſt and third terms being conſtant, the ſecond varies as the fourth. Alſo, as the plane acts againſt the fluid at the angle *CAB*, let *AC* be taken to repreſent the whole force of the plane acting againſt the fluid upon ſuppoſition that no part thereof was loſt, which force would be conſtant, the velocity of the plane being uniform; then, by the reſolution of motion, the force acting perpendicular to the plane will be in proportion to *BC*, or to the ſine of *BAC*, for *AC* : *BC* :: rad. = 1 : ſin. *BAC*, where the firſt and third terms being conſtant, the ſecond varies as the fourth. Hence, the whole action of the plane againſt the fluid in the direction *BC* (being in proportion to the whole quantity of fluid which oppoſes it's motion, and it's effect in the direction *BC* conjointly) will be as ſin. *BAC* × ſin. *BAC*, or as ſin. $\overline{BAC}|^2$. And as action and reaction are equal and contrary, the action of the fluid againſt the plane in the direction *CB*, or the reſiſtance of the fluid, muſt vary in the ſame ratio.



PROP.

PROP. XXVI.

The reſiſtance of the ſame fluid to oppoſe the plane in the direction of it's motion varies as $\overline{\text{ſin } BAC}|^3$, *ſuppoſing, after the reſolution of the reaction of the fluid in the direction CA, into CB and BA, the part BA to be entirely loſt, and CB to take effect.*

62. As the whole effect of the reſiſtance of the fluid upon the plane is that part which is perpendicular to it, let CB repreſent that whole reſiſtance, and reſolve it into CD, DB, then will CD repreſent the reſiſtance which oppoſes the motion of the body; now, rad. $= 1 : \text{ſin. } DBC$, or $\text{ſin. } BAC, :: CB : CD = CB \times \text{ſin. } BAC$; but CB, as repreſenting the whole reſiſtance in the direction CB, varies as $\overline{\text{ſin. } BAC}|^2$ (Art. 61); hence, CD varies as $\overline{\text{ſin. } BAC}|^3$.

By experiments on plane bodies moving both in air and water, I find that they are not reſiſted according to the laws here deduced. Part of the difference may probably be owing to the two latter cauſes mentioned in Art. 57, but it principally ariſes from the force, after reſolution, not taking effect as here ſuppoſed, that part which is parallel to the plane not being all loſt. But the further conſideration of this ſubject falls not within the plan of this work, which is intended only to be an elementary treatiſe.

PROP. XXVII.

The ſame ſuppoſition being made, the reſiſtance of the plane, in a direction perpendicular *to that of it's motion, varies as* $\overline{\text{ſin. } BAC}|^2 \times \text{coſ.} BAC$.

63. For by the laſt Art. DB will repreſent that part of the whole reſiſtance which acts perpendicucarly to the direction of the motion; hence, rad. $= 1 : BCD$, or coſ. $BAC, :: CB : DB = CB \times$ coſ. BAC; but CB, as repreſenting the whole effective part of the force, varies as $\overline{\text{ſin. } BAC}|^2$ (Art. 61); therefore, DB varies as $\overline{\text{ſin. } BAC}|^2 \times$ coſ. BAC.

64. If inſtead of ſuppoſing the plane to move in the fluid, we ſuppoſe the plane to be at reſt and the fluid to move againſt it, the action of the fluid againſt the plane will be juſt the ſame as it's reaction when the plane moves. Hence, the laſt article will ſhow the effect which the wind has upon the ſails of a wind-mill, when at reſt, to put them in motion, admitting our hypotheſis, reſpecting the efficacious part of the force, to be true.

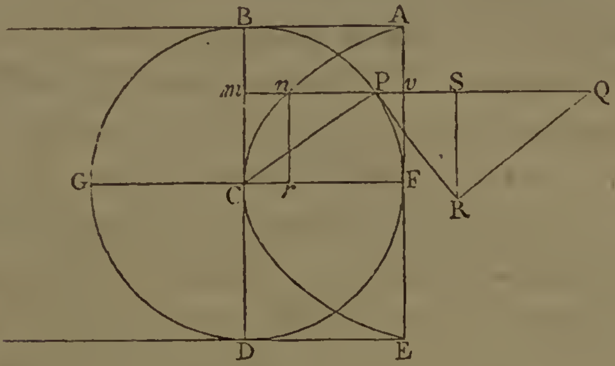
65. If, in the three laſt articles, the area of the plane, the velocity and denſity of the fluid be not given, then (for the reaſons in Arts. 58, 60) the reſiſtance will vary in the above ratios, and the area of the plane, ſquare of the velocity and denſity conjointly.

PROP. XXVIII.

The ſame ſuppoſition being made, let a cylinder, moving in the direction of it's axis, and a ſphere of the ſame diameter, move in the ſame fluid with the ſame velocity; then will the reſiſtance to the motion of the cylinder be double to that of the globe.

66. Let AFE be a diameter of the end of a cylinder, parallel and equal to BD a diameter of the ſphere $BFDG$ whoſe center is C, and CF, DE, BA perpen-

dicular to AE; draw QP parallel to CF the axis of the cylinder, and let it repreſent the force with which a particle of fluid would act perpendicularly at v, the end of the cylinder, in which caſe no part is loſt. Now conceiving the the ſame particle to act upon



the globe at P, part of it's effect will be loſt by the obliquity of the ſtroke; draw PR a tangent to the ſphere in the plane PCF, and QR perpendicular to it; draw alſo RS perpendicular to QP, and produce QP to m. Reſolve the whole force QP into RP, QR, then we here ſuppoſe that the part RP to be wholly loſt, and the part QR only to be effective; reſolve this into QS, SR, then QS is employed in oppoſing the motion of the globe, and SR (being perpendicular to it) can have no effect in that reſpect, and moreover it will be deſtroyed by an equal and oppoſite force, acting at a point, equidiſtant from F, on the other ſide. Hence the force with which the point v at the end of the cylinder is retarded : the force with which the correſponding point P on the globe is retarded :: $QP : QS$ :: (becauſe $QP : QR :: QR : QS$) $QP^2 : QR^2$ :: (by ſim. trian. PRQ, PmC) $PC^2 : Pm^2 :: PC : \frac{Pm^2}{PC}$ :: (taking $vn =$

Pm^2

$\frac{Pm^2}{PC}\Big)\ PC : vn :: vm : vn$; conſequently the whole reſiſtance on the cylinder : that on the globe :: the *ſum of all the* vm's : the *ſum of all the* vn's. Draw nr parallel to mC. Now $vm : vn :: PC^2 : Pm^2$, therefore vm, or CF, : mn, or Cr, :: PC^2, or AF^2, : $PC^2 - Cm^2$, or rn^2, the locus therefore $AnCE$ of all the points n is a parabola. Conceive now this whole figure to revolve about FCG, then will AE generate the end of the cylinder, and BFD that half of the globe which oppoſes itſelf to the medium, or which is reſiſted; alſo, the ſum of all the vm's will be the ſolid generated by the parallelogram $AEDB$, and the ſum of all the vn's will be the ſolid generated by the inſcribed parabola ACE, which ſolids (Fluxions, pag. 78) are as 2 : 1; hence, the reſiſtance of the cylinder : the reſiſtance of the globe :: 2 : 1.

It appears by experiment, that this propoſition is not true when bodies move either in air or water, the reſiſtance of the globe, compared with that of the cylinder, being leſs than that which the theory gives it.

Prop. XXIX.

The ſame ſuppoſition being admitted, if a globe whoſe diameter is d *move in a reſiſting medium whoſe denſity is* n, *with a velocity* v, *the reſiſtance will vary as* $v^2 d^2 n$.

67. For the reſiſtance of a globe is (Art. 66) equal to half the reſiſtance of the baſe of a cylinder of the ſame diameter, moving in the direction of it's axis with the ſame velocity; therefore the reſiſtance

of the globe varies as the reſiſtance of the cylinder; but the end of the cylinder being a circle, whoſe area is as d^2, the reſiſtance (Art. 60) varies as v^2d^2n; therefore the reſiſtance of the globe varies as v^2d^2n.

If the Reader wiſh to ſee any thing further upon the reſiſtance of bodies moving in fluids, he may conſult the *Fluxions*, page 125.

Prop. XXX.

As a body deſcends in a fluid, it continually adds more weight to the fluid until it has acquired it's greateſt velocity, at which time, the weight added to the fluid from the reſiſtance is equal to the relative weight of the body.

68. For as long as the velocity of the body increaſes, the action of the body upon the fluid will continue to increaſe; and when the body has acquired it's greateſt velocity, the reſiſtance becomes equal to the weight of the body in the fluid, and the body then acts againſt the fluid with it's whole relative weight.

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SECTION IV.

ON THE TIMES OF EMPTYING VESSELS, AND ON SPOUTING FLUIDS.

Prop. XXXI.

IF a fluid run through any tube, kept continually full, and the velocity of the fluid in every part of the ſame ſection be the ſame, the velocities in different ſections will be inverſely as the areas of the ſections.

69. For the ſame quantity of fluid runs through every ſection in the ſame time; now the quantity running through any ſection (A) with the velocity (V), in any given time, is manifeſtly in proportion to A and V conjointly, or to $A \times V$; and as this quantity is conſtant, $A \times V$ is conſtant, conſequently V varies inverſely as A.

70. Cor. Hence, the velocity of water in a river increaſes as the breadth and depth decreaſe, the rule however cannot be applied with accuracy here, as the water at the bottom, from it's unevenneſs, cannot

move with the ſame velocity as at the top; and where the riſe and fall at the bottom is very quick, there is probably a great deal of ſtagnant water.

PROP. XXXII.

Let a veſſel be kept filled with a fluid whilſt it continues to run out at an orifice; and let the quantity run out in a given time, and the area of the orifice be given, to find the velocity at the orifice.

71. Let a = the area of the orifice, m = the quantity run out in the time t'', and v = the velocity. Conceive all the fluid which runs out to form a cylinder whoſe baſe is a, and length l; then $a \times l = m$, hence $l = \frac{m}{a}$; therefore, in the time t'' the fluid, with the firſt velocity v, would have deſcribed the ſpace $\frac{m}{a}$; hence, $t'' : 1'' :: \frac{m}{a} : v = \frac{m}{at}$ the velocity in a ſecond at the orifice.

PROP. XXXIII.

If a fluid run out from the bottom or ſide of a veſſel, and the area of the orifice be very ſmall when compared with the bottom, the velocity at the orifice is that which a body would acquire in falling through a ſpace equal to half the altitude of the fluid above the orifice, very nearly.

72. When a fluid iſſues from a veſſel, the water ruſhing towards the orifice in all directions cauſes a contraction in the ſtream; and at a diſtance from the orifice equal to it's diameter, Sir I. NEWTON meaſured

the diameter of the ſection of the ſtream (which ſection he called the *vena contracta)*, and found it to be to the diameter of the orifice as 21 : 25. Now the area of the orifice : the area of the vena contracta (they being ſuppoſed to be ſimilar) :: $25^2 : 21^2$, which is very nearly as $\sqrt{2} : 1$; hence, as (Art. 69) the velocity is inverſely as the area of the ſection, the velocity at the vena contracta : the velocity at the orifice :: $\sqrt{2} : 1$. Alſo from the quantity of water running out in a given time, and the area of the vena contracta, Sir I. NEWTON found (Art. 71) that the velocity at the vena contracta is that which a body acquires in falling down the altitude of the fluid above the orifice; hence, the velocity at the orifice (being leſs than that at the vena contracta in the ratio of $\sqrt{2} : 1$) is (Mech. Art. 241) that which a body would acquire in falling down half the altitude.

73. The principle to be eſtabliſhed, in order to determine the time of emptying a veſſel through an orifice, is the relation between the velocity of the fluid at the orifice, and the altitude of the fluid above it. Moſt writers upon this ſubject have conſidered the column of fluid over the orifice as the expelling force, and from thence ſome have found the velocity at the orifice to be that which a body would acquire in falling down the *whole* depth of the fluid; and others, that it is ſuch as is acquired in falling through *half* the depth; and this without regard to the magnitude of the orifice; whereas it is manifeſt from experiment, that the velocity at the orifice, the depth of the fluid being the ſame, depends upon the proportion which the magnitude of the orifice bears

to the magnitude of the bottom of the veſſel. Concluſions thus contrary to matter of fact ſhow, either that the principle aſſumed is not true, or that it is not applicable to the preſent caſe. The moſt celebrated theories upon this ſubject are thoſe of D. BERNOULLI and M. D'ALEMBERT; the *former* deduced his concluſions from the principle of the *conſervatio virium vivarum*, or, as he calls it, the *equalitas inter deſcenſum actualem aſcenſumque potentialem*, where by the *deſcenſus actualis* he means the actual deſcent of the center of gravity, and by the *aſcenſus potentialis* he means the aſcent of the center of gravity, if the fluid which flows out could have it's motion directed upwards; and the *latter*, from the principle of the *equilibrium* of the fluid. This principle of M. D'ALEMBERT leads immediately to that aſſumed by D. BERNOULLI, and conſequently they both obtain the ſame fluxional equation, the fluent of which expreſſes the relation between the velocity of the fluid at the orifice, and the perpendicular altitude of the fluid above it. How far the principles here aſſumed can be applied in our reaſoning upon fluids, can only be determined by comparing the concluſions deduced from them with experiments.

The general fluxional equation above mentioned cannot be integrated, and therefore the relation between the velocity of the fluid at the orifice and it's depth cannot from thence be determined in all caſes. If the magnitude of the orifice be indefinitely leſs than that of the ſurface of the fluid, the equation gives the velocity of the fluid equal to that which a body would acquire by falling *in vacuo* through a ſpace equal to the depth of the fluid. But the velocity here determined is not that at the orifice, but at the vena contracta; for the fluid by flowing in all directions to the orifice contracts the ſtream, and the velocity being inverſely as the area of the ſection, the velocity continues to increaſe as long as the ſtream, by the expelling force of the fluid, continues to decreaſe, and when the ſtream ceaſes to be contracted by that force, at that ſection of the ſtream, or at the vena contracta; the velocity is found, by this theory, to be that which a body would acquire in falling through a ſpace equal to the depth of the fluid. To determine therefore, by theory, the time in which a veſſel empties itſelf, we muſt know the proportion between the area of the orifice and the area of the vena contracta; but no theory will give this. The times therefore of emptying veſſels, even in the moſt ſimple caſe, cannot be determined by theory alone.

PROP. XXXIV.

If a veſſel empty itſelf through a very ſmall orifice, the velocity of the fluid at the orifice varies as the ſquare root of the altitude (a) of the fluid above it.

74. For the velocity at the orifice is that which is acquired (Art. 72) in falling down $\frac{1}{2}a$, and conſequently (Mech. Art. 241) it varies as $\sqrt{\frac{1}{2}a}$, or as $\sqrt{a}$.

PROP. XXXV.

If a veſſel empty itſelf through an orifice at the bottom, and the area of the ſection, parallel to the bottom, be every where the ſame, the velocity of the ſurface of the fluid is uniformly retarded.

75. For (Art. 69) the velocity of the deſcending ſurface is to the velocity at the orifice as the area of the orifice to the area of the ſurface, which is a conſtant ratio; hence, the velocity of the deſcending ſurface varies as the velocity at the orifice, or as $\sqrt{a}$ by the laſt article; that is, the velocity of the deſcending ſurface varies as the ſquare root of the ſpace which it has to deſcribe, which is exactly the caſe of a body projected perpendicularly from the Earth's ſurface, where (Mech. Art. 248) the velocity is as the ſquare root of the ſpace to be deſcribed; and as the retarding force is conſtant in the latter caſe, it muſt alſo be conſtant in the former.

Prop. XXXVI.

If a cylindrical or priſmatic veſſel, having an orifice at the bottom, be kept conſtantly full, twice the quantity which the veſſel contains will run out in the time it would have emptied itſelf.

76. For the ſurface of the fluid being uniformly retarded, and it's velocity becoming equal to nothing at the bottom, the ſpace (Mech. Art. 237) which the ſurface *would* deſcribe with the firſt velocity continued uniform for the time in which the veſſel would empty itſelf is double the ſpace which the ſurface actually *does* deſcribe in the time it empties itſelf; in that time therefore the quantity diſcharged in the former caſe is double that in the latter, becauſe the quantity run out when the veſſel is kept full may be meaſured by what *would* be the deſcent of the ſurface, if it could deſcend with it's firſt velocity.

PROP. XXXVII.

If a cylindrical or priſmatic veſſel, whoſe altitude is h, *empty itſelf through a very ſmall orifice (*a*) at the bottom (*A*), the time (*t*) of emptying itſelf* $=,3526\times\frac{A}{a}\times\sqrt{h}$.

77. By the laſt Propoſition, in the time t the quantity diſcharged with the firſt velocity (v) is equal to $2A\times h$; hence, (Art. 71.) $v=\frac{2A\times h}{a\times t}$; therefore, $t=\frac{2A\times h}{a\times v}$; but (Mechanics, Prop. 248, and Art. 72.) if $s=16\frac{1}{12}$ feet, $v=\sqrt{4s\times\frac{1}{2}h}=\sqrt{2sh}$; conſequently, $t=\frac{A}{a}\times\frac{2h}{\sqrt{2sh}}=\sqrt{\frac{2}{s}}\times\frac{A}{a}\times\sqrt{h}=,3526\times\frac{A}{a}\times\sqrt{h}$.

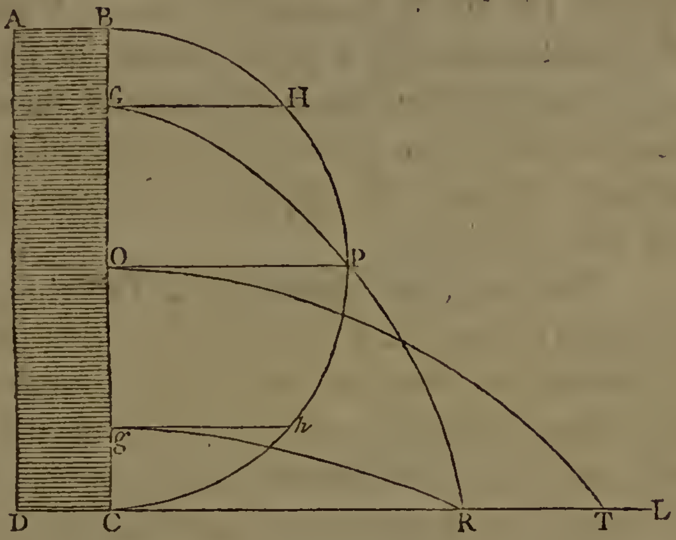
59. Cor. Hence, the time of emptying any other depth k from the bottom is $,3526\times\frac{A}{a}\times\sqrt{k}$; conſequently, the time of emptying any depth $h-k$ from the top $=,3526\times\frac{A}{a}\times\overline{\sqrt{h}-\sqrt{k}}$.

For the time of emptying veſſels in general, ſee the Fluxions, page 217.

PROP. XXXVIII.

To find the diſtance to which fluids will ſpout from the ſide of a veſſel placed upon an horizontal plane.

78. Let $ABCD$ be a veſſel filled with a fluid, and BC be perpendicular to the horizontal plane CL, and upon BC let a ſemicircle be deſcribed, and GH an ordinate perpendicular to BC; then the diſtance CR to which the fluid ſpouts through a very ſmall orifice at G is $=2GH$. By Art. 72 the velocity at the vena contracta, which is extremely near to the veſſel, is that



which a body would acquire in falling down BG; we are therefore to conſider this as the velocity with which the fluid is projected, and not the velocity at the orifice. Now (Mech. Art. 313, and 320) the curve GR deſcribed by the fluid is a parabola, and BG is one fourth of the parameter belonging to the point G, which point is the vertex of the parabola, the fluid ſpouting out horizontally; hence GC is the abſciſſa and CR it's ordinate, and by the property of the parabola, $4BG \times GC = CR^2$, therefore $CR = 2\sqrt{BG \times GC}$ $= 2GH$, by the property of the circle.

79. Cor. If $Cg = BG$, then $gh = GH$, and the

fluid ſpouts to the ſame diſtance from g as from G. If BC be biſected in O, then the diſtance CT, to which the fluid ſpouts, is equal to $2OP=BC$, and this is the greateſt diſtance, OP being the greateſt ordinate.

80. If the fluid ſpout perpendicularly upwards, it ought (Art. 72) to riſe to the altitude of the ſurface of the water in the veſſel; but it falls a little ſhort of this, partly from the reſiſtance of the air, and partly from the falling back of the water. If the water ſpout upwards through a pipe, inſtead of ſimply an hole, it does not aſcend ſo high, becauſe there being no vena contracta, the velocity is not increaſed immediately after it leaves the pipe, as it is when it flows out of a ſimple orifice.

81. The very near agreement of this theory with experiments proves, very ſatisfactorily, that the velocity of projection muſt be that which is acquired in falling down BG, the whole altitude of the fluid. And the agreement of the theory of emptying veſſels with experiments ſhows very clearly that the velocity *at* the orifice muſt be that which is acquired in falling through half the altitude of the fluid. Almoſt immediately, therefore, after the fluid gets out at the orifice, it's velocity is increaſed in the ratio of $1 : \sqrt{2}$.

SECTION V.

ON THE ATTRACTION OF COHESION, AND ON CAPILLARY ATTRACION.

PROP. XXXIX.

IF two globules of mercury, lying on a ſmooth plane, be brought to touch, an attraction then takes place, and they immediately ruſh together and form one complete globule.

82. If the globules be examined with a microſcope, no effect is found to take place till they are actually in contact, and then they ruſh violently together; this therefore can only be accounted for by an attraction which begins at the inſtant they come into contact.

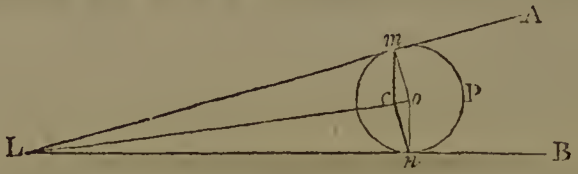
The ſame effect takes place between two globules of water, when they are laid on a ſurface on which they can freely move.

PROP.

PROP. XL.

If two glaſs planes AL, BL, (of which BL is horizontal) be inclined at a very ſmall angle, and juſt moiſtened with oil, and a drop P of oil be placed between them, it will move towards their concourſe L.

83. This ariſes from the attraction of coheſion between the drop and the glaſs planes; for if *o* be the center of the drop, and *om*, *on* repreſent the attraction



of the drop to each plane, the whole attraction acting perpendicular to the planes, then the compound motion *oc*, being directed to *L* will give the drop a motion towards that point. The planes are firſt moiſtened, that the drop may move freely.

84. If the point *L* be elevated, until the motion of the drop ceaſes, the action *oc* then becomes equal to the accelerative force of *P* down the inclined plane *LB*; conſequently the ratio of the accelerative force upon that inclined plane to the force of gravity, or (Mech. Prop. 61.) the height of the plane to the length, gives the ratio of the attraction *oc* to the weight of the drop.

PROP. XLI.

If two plane ſurfaces of metal, &c. be ſmeared with oil, greaſe, &c. and preſſed together, they will cohere very ſtrongly.

85. This effect ariſes from two cauſes, the preſſure of the ſurrounding air, in conſequence of the air from between being expelled, and from the attraction of coheſion. That it ariſes partly from the former cauſe, is manifeſt from hence, that if two plates be thus put together and cohere pretty ſtrongly in the air, when they be ſuſpended in the receiver of an air pump, after exhauſting the air, the under plate will frequently fall; when it does not fall, it ſhows that there muſt be an attraction of coheſion, at leaſt equal to the weight of that plate. If the air be expelled by different ſubſtances, as oil, turpentine, greaſe, &c. it is found that the attraction of coheſion is different. This therefore muſt ariſe, either from the air being expelled more perfectly by one than the other, or that the attraction is rendered ſtronger by one than by another.

It is this attraction of coheſion by which the conſtituent particles of a body, admitting them to be in contact, are kept together. When you break a body, you overcome this attraction, and if you could join the parts together again exactly in the ſame manner, it would be as ſtrong as before. On this principle we may explain the different degrees of hardneſs of bodies. Hard bodies may conſiſt of conſtituent particles which touch in a great part of their ſurfaces, and thus their attraction may be very great. The conſtituent particles of ſoft bodies may touch in a few points, and thus their attraction will be weakened. Solids are ſuppoſed to be diſſolved in menſtruums, from the attraction of coheſion between the particles of the fluid and body being greater than the attraction between the conſtituent particles of the ſolid.

PROP.

PROP. XLII.

There is an attraction of coheſion between water and glaſs.

86. For take a piece of very clean glaſs, and hold it in an horizontal poſition, and a drop of water will remain ſuſpended from its under ſide.

PROP. XLIII.

The conſtituent particles of water attract each other.

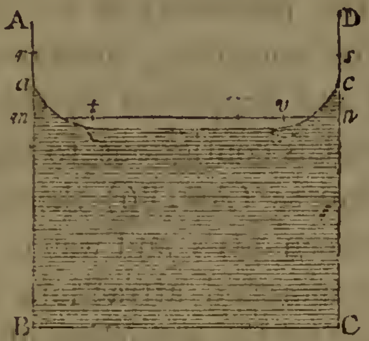
87. For a ſmall quantity of water always forms itſelf into a globule, if no external circumſtance tend to prevent it; this can ariſe only from the mutual attraction of it's parts. If the quantity be large, the upper ſurface will grow flat, from it's gravity overcoming the attraction of it's parts, but the ſides will ſtill be convex.

PROP. XLIV.

There is an attraction between water and glaſs, at a diſtance.

88. Pour water into the glaſs veſſel *ABCD*, and if no attraction took place between the water and glaſs, the ſurface *mn* would be horizontal. Take $mr = mt = ns = nv$, and ſuppoſe the glaſs to act upon the water through theſe diſtances; then the glaſs *mr* will attract the ſurface *mt*, and *ns* will attract *nv*, and part of this attraction acting upwards, the gravity

of the columns of water under mt, vn will be diminiſhed by the attraction, and more diminiſhed the nearer to the ſides, conſequently their length muſt be increaſed in order to be in equilibrium with the other columns whoſe gravity is not diminiſhed; hence the water will riſe in a curve ax, cz from the points x and z as far as the attraction extends, and the other part xz will be horizontal; now when water is put into a glaſs veſſel, the ſurface of the fluid puts on the form $axzc$; we conclude, therefore, that glaſs acts upon water at a diſtance. In like manner, if any piece of glaſs be immerſed in water, the water will riſe on each ſide of the glaſs.



89. Cor. 1. Hence, if two pieces of glaſs, parallel to each other, be immerſed in the veſſel, the water will riſe againſt each. Let them be ſo near, that the two curves ax, cz may juſt meet, then will a certain quantity of the fluid between them be raiſed above the ſurface of the fluid in the veſſel. Bring them nearer, and as the glaſs ſtill exerts the ſame attraction, it muſt, upon this principle, raiſe the ſame quantity of water, and therefore the altitude will be inverſely

as the diſtance of the planes; for the ſame attraction being exerted, there muſt be the ſame quantity of fluid ſupported, conſequently the altitude of the fluid will increaſe as the proximity of the planes decreaſe. This ſeems to be concluſive, admitting the water to be both raiſed and ſupported by the attraction of a ſmall part of the glaſs contiguous to the upper ſurface of the fluid. But as (Art. 86) there is an attraction of coheſion between glaſs and water, after the water is raiſed by the aforeſaid attraction, it will then, in part be ſupported by the attraction of coheſion. An additional quantity of water may therefore probably be *ſupported* from this cauſe.

90. Cor. 2. If the glaſs planes be *inclined* to each other, then it follows, from theſe principles, that as the diſtance between the glaſſes decreaſes, the altitude of the water will increaſe. Mr. HAUKSBEE informs us, that he very accurately meaſured the abſciſſas and ordinates of the curve formed by the upper ſurface of the water between the glaſs planes, and concluded it to be the common hyperbola, having the ſurface of the fluid, and concourſe of the planes, for it's aſymptotes. Now if we admit that the water is raiſed, and alſo ſupported, by the attraction of the glaſs lying juſt above the ſurface of the water, the curve ought to be the common hyperbola; for if we divide the water into laminæ of the ſame thickneſs, then there being the ſame attraction exerted upon each, the ſame quantity will be ſupported, and therefore the altitudes (or ordinates) muſt be inverſely as the lengths of the laminæ, or diſtances (abſciſſas) of the laminæ from the concourſe of the planes, which is

the property of the hyperbola between the above mentioned aſymptotes.

PROP. XLV.

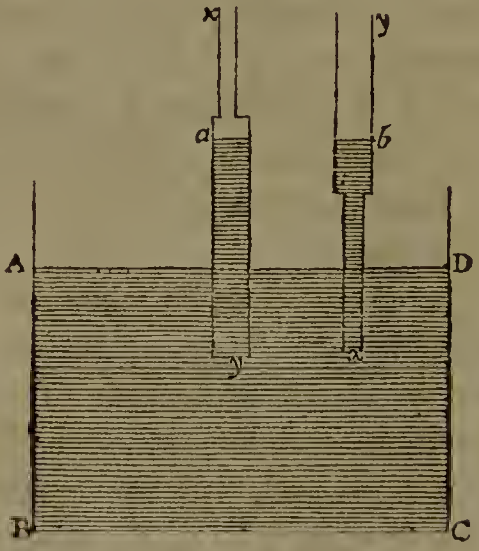
If very ſmall glaſs tubes, called capillary *tubes, be dipped in water, the water is found to ſtand in them, above the level of that in the veſſel, at altitudes which are either accurately or nearly in the inverſe ratio of their diameters.*

91. If we admit the fluid to be raiſed and ſupported only by an annular ſurface of the glaſs contiguous to the upper ſurface of the water, the ratio ought to be accurately ſo. For let $d =$ the diameter of the tube, $a =$ the altitude of the water in it; then the breadth of the attracting annulus being conſtant (it being the diſtance to which the attraction of the glaſs reaches), the area of this annulus, or the attracting ſurface, will be in proportion to the circumference of the tube, and conſequently to the diameter d; alſo the quantity of attraction muſt be in proportion to the quantity of water ſupported, or to $d^2 \times a$, the tube being cylindrical; hence, $d^2 \times a$ varies as d, therefore $d \times a$ is conſtant, and conſequently a varies inverſely as d. Experiments ſhow that this concluſion is accurately, or very nearly, true.

92. Cor. 1. If the tube be taken out of the fluid and laid in an horizontal ſituation, the fluid will recede from that end which was immerſed. For at that end there is no attracting annulus beyond the fluid, whereas there is at the other end, and conſequently the fluid will be drawn towards the empty part of the tube,

until the length of the other end, left free from the fluid, be equal to the diſtance to which the attraction reaches.

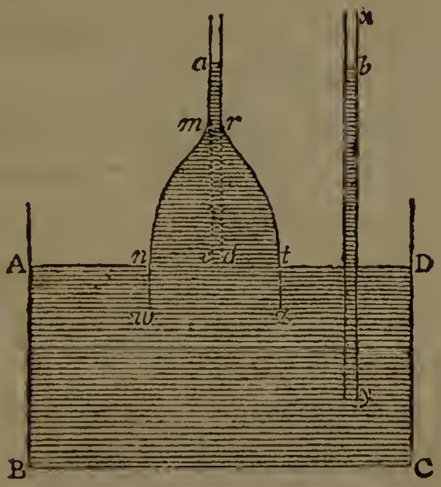
93. Dr. HAMILTON, in his Eſſays, thinks, that the fluid is not ſupported by the attraction of the annular ſurface of the tube contiguous to the upper ſurface of the water, but by the annulus at the bottom, contiguous to the bottom of the tube; this he ſuppoſes will firſt draw up a plate of water immediately under it, and then a ſucceſſion of plates, till the weight of the whole is equivalent to the attraction of that annulus. If this were the caſe, the quantity ſupported, and conſequently the altitude of the fluid, would depend upon the orifice at the botton, whereas, experiments ſhow that the altitude at which the fluid is ſupported depends upon the diameter of the tube at the upper ſurface of the fluid, without any regard to the form of the tube below it. See other objections to this ſolution in Mr. PARKINSON's *Hydroſtatics*, page 40.



94. Dip the tube xy (being two cylindrical tubes

joined together) into water, and ſuppoſe the fluid to riſe to *a* by the capillary attraction; invert the tube, and the fluid will riſe to *b*, the ſame altitude where the diameter is the ſame. If we admit the principle in Art. 91, the annular ſurface being the ſame in both, the ſame quantity of fluid ought to be ſupported, whereas, when the ſmall end is downwards, the quantity ſupported is leſs. But if we admit the fluid to be partly retained by the attraction of coheſion, the quantity of the ſurface being leſs with the ſmaller end downwards, the whole ſupport is leſs, and therefore the quantity ſupported ought to be leſs. This ſeems to be in favour of the ſupport being partly by the attraction of coheſion.

95. If the water riſe in the capillary tube *xy* to *b*, and another veſſel *az* be put into the water, having the upper part capillary, and equal to that of *xy*, but the lower part of any ſize; then if the air be drawn out



of

of this veſſel by ſuction until the water enters into the capillary part, it will ſtand at the ſame altitude as in the tube *xy*, after the ſuction ceaſes and the air is admitted into the capillary part. Here the cylindrical part *ad* is ſupported by the ſame power as the water in the other tube, and the other part *mvn*, *rtd* is ſupported by the preſſure of the air upon the ſurface of the water in the veſſel, in conſequence of the air being drawn from within; and this is proved from hence, that if the whole be put under a receiver of an air pump, and the air be exhauſted from the ſurface of the veſſel, the water will not be ſupported. Dr. JURIN ſays, that the water will be ſupported in vacuo; but in his time, the air pumps would not exhauſt ſufficiently to determine this point, for the altitude to which the water riſes being very ſmall, it requires a conſiderable degree of exhauſtion before the water will fall. The pumps which are now made, ſhow that the water will not be ſupported after a conſiderable degree of exhauſtion.

96. If a veſſel of water be put under the receiver of an air pump, and the air be exhauſted, and capillary tubes be then immerſed in it, the water will riſe to the ſame altitude as when the veſſel was expoſed to the air; the air, therefore, has nothing to do in cauſing the aſcent of the water.

97. Different fluids will riſe to different altitudes in the ſame tube. Spirituous liquors, which are lighter than water, riſe to a leſs height than water, which, of all fluids, appears to riſe to the greateſt height. This can ariſe only from the different degrees of attraction of theſe fluids to glaſs.

98. The diameter of the tube multiplied into the altitude of the fluid in it is (Art. 91) accurately or very nearly a conſtant quantity, which, by experiment, is found to be ,053 of an inch.

99. The diameter of a capillary tube is thus found to a very great degree of accuracy. Put into the tube ſome mercury whoſe weight in grains is w, and let it occupy a length of the tube l; then if 13,6 be the ſpecific gravity of mercury (which it is when pureſt) that of water being 1, the diameter $= \sqrt{\frac{w}{l}} \times ,01923$. For if $d =$ the diameter, the content of the mercury $= d^2 l \times ,7854$, and as one cubic inch of mercury weighs 3443 grains, we have $1 : 3443 :: d^2 l \times ,7854 : w$, wence $d = \sqrt{\frac{w}{l}} \times ,09123$. This rule is given by Mr. ATWOOD in his *Analyſis*.

PROP. XLVI.

There is a ſmall attraction of coheſion between mercury and glaſs.

100. For a very ſmall globule of mercury will adhere to the under ſide of a clean piece of glaſs.

PROP. XLVII.

There is a ſtrong attraction of the conſtituent particles of mercury towards each other.

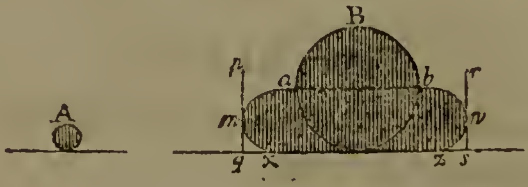
101. For a ſmall quantity of mercury laid upon a piece of glaſs will, as to ſenſe, form itſelf into a per-

fect ſphere. If two of theſe ſpheres be brought into contact, at the inſtant they touch they will ruſh together with a ſurpriſing quickneſs, and form a ſingle ſphere. This can only be explained upon the principle of the mutual attraction of all the parts. As you increaſe the quantity of mercury, it will begin to deviate from a perfect ſphere, and grow flatter on the upper ſide, ariſing from the gravity of the mercury becoming ſenſibly greater than the mutual attraction of it's conſtituent particles; and when the quantity becomes conſiderable, the upper ſurface will not ſenſibly differ from a perfect plane, but the ſides will retain their convexity.

Prop. XLVIII.

If mercury be put into a glaſs veſſel, it will ſtand loweſt at the ſides, and riſe in a curve till the ſurface becomes, as to ſenſe, a plane.

102. This ariſes from the mercury attracting itſelf by a greater force than it is attracted to the glaſs, and may be thus explained. A very ſmall quantity *A* of mercury laid upon glaſs, will, as to ſenſe, form itſelf into a perfect ſphere. If we take a large quantity *B*, it will



not preſerve it's ſpherical form, the force of gravity deſtroying that figure by overcoming the mutual attraction of the particles, and the mercury will put on the form *xmabnz*; and if two pieces of glaſs *pq*, *rs* be made to touch the mercury at *m* and *n*, the form will not be ſenſibly altered. If therefore we take a glaſs veſſel *pqsr* and put mercury into it, the upper ſurface will ſtill be in the form *mabn*, for it can manifeſtly make no difference whether we put the glaſs to the mercury, or the mercury to the glaſs; except that in the latter caſe, the ſpaces *mqx*, *nsr* will be filled with mercury, which can have no effect upon the upper ſurface.

103. Cor. 1. Hence, if a piece of glaſs be dipped into mercury, the mercury will be depreſſed on each ſide of the glaſs, in the ſame manner.

104. Cor. 2. If the two pieces of glaſs, *pq*, *rs* be brought ſo near, that the depreſſed parts of the mercury may meet, the mercury will be depreſſed to a diſtance which is inverſely as the diſtance of the glaſſes, accurately or nearly ſo.

105. If ſmall capillary tubes be put into a veſſel of mercury, the fluid in the tubes will be depreſſed to diſtances below the ſurface of the fluid in the veſſel, which are found, by menſuration, to be inverſely as their diameters, accurately or nearly ſo.

106. If two glaſs planes, inclined at a ſmall angle, be put into a veſſel of mercury, the mercury between them will be depreſſed below the ſurface of the mercury without the planes, and that depreſſion is found (as nearly as it can be determined by menſuration) to be inverſely as the diſtance from the concourſe of the

planes,

planes, and therefore the curve is an hyperbola, having the concourſe of the planes for one aſymptote, and the ſurface of the mercury againſt the planes without, for the other aſymptote.

* * *

SECTION VI.

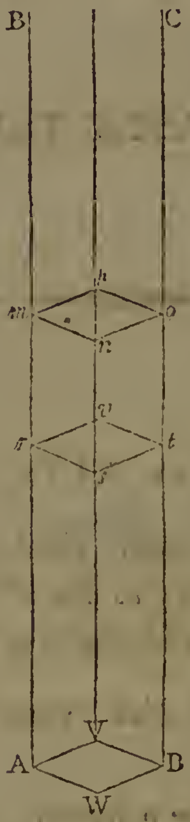
ON ELASTIC FLUIDS.

Prop. XLIX.

If the particles of an elaſtic fluid repel each other with forces varying inverſely as the n^{th} *power of their diſtances, and d repreſent the denſity of any part, and c the compreſſive force upon it, then c varies as* $d^{\frac{n+2}{3}}$.

107. Let *ABCD* be a ſquare column of the fluid, *mnop*, *rstv* two ſections parallel to the baſe (and conſequently each a ſquare equal to the baſe) whoſe diſtance *mr* is equal to *mn*, one ſide of the ſquare, then will the fluid contained between theſe ſections be a cube. Let *d* be the denſity of the fluid in this cube, ſuppoſed to be indefinitely ſmall, *c* the compreſſive force on it, and *r* the diſtance of the particles; then one ſide of the cube being indefinitely ſmall, *d*, *c* and *r* may be conſidered as the ſame for the whole of this cube. Now the number of particles

in *mn* is as $\frac{1}{r}$, conſequently the number in the whole ſection *mnop* is as $\frac{1}{r^2}$. Alſo the repulſive force of all the particles in *mnop*, being as the number of particles and force of each conjointly, is as $\frac{1}{r^2} \times \frac{1}{r^n} = \frac{1}{r^{n+2}}$; and as the repulſive force of each particle acts in every direction, this repulſive force acting upwards muſt be equal to the compreſſive force which it ſuſtains, or *c* will vary as $\frac{1}{r^{n+2}}$; they will not neceſſarily be equal, becauſe



becauſe $\frac{1}{r^{n+2}}$ does not repreſent the *quantity* of the repulſive force, only what it is proportional to. Alſo the number of particles in the cube is as $\frac{1}{r^3}$, and therefore (Art. 4) d varies as $\frac{1}{r^3}$, hence $d^{\frac{1}{3}}$ varies as $\frac{1}{r}$, and $d^{\frac{n+2}{3}}$ varies as $\frac{1}{r^{n+2}}$; but a varies as $\frac{1}{r^{n+2}}$; conſequently c varies as $d^{\frac{n+2}{3}}$.

108. It appears by experiment, that the compreſſive force of the atmoſpheric air varies as the denſity, or c varies as d, therefore in this caſe $\frac{n+2}{3}=1$, and hence $n=1$, conſequently the particles of air repel each other with forces which vary inverſely as their diſtances. Alſo, as the compreſſive force of air is equal to it's elaſtic force, theſe balancing each the other, the elaſtic force muſt vary as the denſity.

109. It is manifeſt that there can be no fluid whoſe denſity varies in any inverſe ratio of the compreſſive force, that is, you can never, by increaſing the compreſſive force, diminiſh the denſity, as any increaſe of the compreſſive force muſt compreſs the fluid into a leſs ſpace, and therefore increaſe the denſity, unleſs the particles of the fluid were abſolutely in contact, in which caſe the denſity would remain the ſame under any preſſure, which is probably not the caſe with any fluid. Hence $n+2$ muſt be always poſitive, that is, n muſt be ſome whole poſitive number, or a negative number leſs than 2, in order to conſtitute a fluid conſiſting of particles which repel each other. If we admit

water to be compreſſible in a very ſmall degree, the particles muſt be kept at a diſtance by ſome repulſive force, and n muſt be very nearly $= -2$; hence, upon this ſuppoſition, the repulſive force of the particles of water varies nearly as the ſquare of their diſtances.

Prop. L.

Air has weight.

110. If a veſſel be exhauſted of air, and balanced at one end of a beam, upon admitting the air the veſſel preponderates. This clearly proves that air has weight, and therefore it muſt preſs upon all bodies; and from the weight neceſſary to balance the veſſel after the air is admitted, compared with the weight of the veſſel of water, we get the ſpecific gravity of air to that of water (Art. 3), which is, as 1 to about 885 in the mean ſtate of the air, or when the barometer ſtands at 29½ inches, according to Mr. Hauksbee. Others have made the ſpecific gravities as 1 to 850, when the barometer ſtands at 30 inches.

LEMMA.

111. If $a : b :: b : c :: c : d ::$ &c. then by Euclid, B. 5. p. 12. $b : c :: b+c+d+$&c. : $c+d+e$ +&c. hence, $a : b :: b+c+d+$&c. : $c+d+e+$&c.; for the ſame reaſon, $b : c :: c+d+e+$&c. : $d+e+f$ +&c. and ſo on to the end of the ſeries. Hence, vice verſâ, if $a : b :: b+c+d+$ &c. : $c+d+e+$&c. and $b : c :: c+d+e+$ &c. : $d+e+f+$ &c. and ſo on, then will $a : b :: b :: c : c :: d : d ::$ &c.

PROP. LI.

If the force of gravity be conſidered as conſtant, and altitudes from the Earth's ſurface be taken in arithmetic progreſſion, the correſponding denſities of the air will decreaſe in geometric progreſſion.

112. Conceive the whole atmoſphere to be divided into an indefinite number of laminæ of *equal* thickneſs, parallel to the Earth's ſurface; and let a, b, c, &c. repreſent the reſpective denſities of theſe laminæ, beginning at the ſurface of the earth; then the compreſſive force on the laminæ a, b, c, &c. will be proportional to the weight incumbent upon each, that is, the ſum of the weights of all the laminæ above; but the weight of each lamina is as the denſity × it's thickneſs, or, as the thickneſs is the ſame, as it's denſity; hence the compreſſive forces on a, b, c, &c. will be as the ſum of all the quantities which repreſent the denſities above them, or as $b+c+d+$&c. $c+d+e+$&c. $d+e+f+$&c. &c. &c. But (Prop. LVII) the compreſſive force of the air is as it's denſity; hence, $a : b :: b+c+d+$&c. $: c+d+e+$&c. and $b : c :: c+d+e+$&c. $: d+e+f+$&c. and ſo on; hence, by the above Lemma, $a : b :: b : c :: c : d :: d : e ::$ &c. Now the laminæ being of the ſame thickneſs, the laſt proportion ſhows that as you aſcend by equal ſpaces, or in arithmetical progreſſion, the denſities decreaſe in geometrical progreſſion.

PROP. LII.

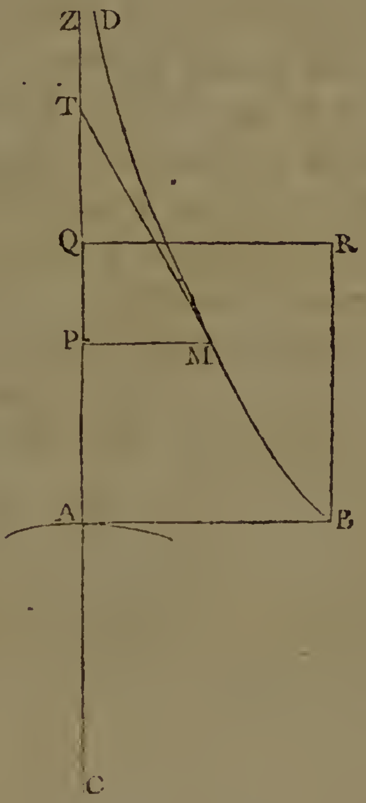
Given the denſity of the air, to find the correſponding altitude; and the converſe.

113. By the nature of logarithms, if the natural numbers be in geometrical progreſſion, their logarithms are in arithmetical progreſſion; hence, as the altitudes increaſe in arithmetical progreſſion whilſt the correſponding denſities of the air decreaſe in geometrical progreſſion (Art. 112), it follows, that the altitudes increaſe as the logarithms of the denſities decreaſe. Hence, if at the altitudes x and y the denſities be m and n times leſs, or be $\frac{1}{m}$ and $\frac{1}{n}$, the denſity of the ſurface being unity, we have $x : y :: \log. \frac{1}{m} : \log \frac{1}{n} :: \log. m : \log. n$. Now Mr. COTES (*Hyd.* p. 103) collected from experiment, that at the altitude of 7 miles, the denſity is 4 times leſs than the denſity at the ſurface; hence, if $y = 7$, $n = 4$, we have, $x : 7 :: \log. m : \log. 4$, therefore $x = 7 \times \frac{\log. m}{\log. 4} = 11{,}626 \times \log. m$; if therefore the denſity $\frac{1}{m}$ be given, we know x. Alſo $\log. m = \frac{x}{7} \times \log. 4$, therefore (Flux. Art. 109.) $m = 4^{\frac{x}{7}}$, conſequently if the altitude be given, the denſity $\frac{1}{m}$ will be known. But the denſity is inverſely as the rarity, that is, if the denſity be 4 times leſs, or be expreſſed by ¼, the rarity will be 4 times greater, or will be expreſſed by 4; hence m may expreſs the rarity of the air, that at the ſurface being unity. This rule is not accurate, becauſe it ſuppoſes the compreſſive

force of the air to be as it's denſity, which is not true, unleſs the temperature be the ſame.

If it ſhould appear that the altitude at which the denſity is 4 times leſs than at the ſurface, be not 7 miles, then 11,626 muſt be altered in the ratio of 7 to that altitude.

114. Let *CA* be the radius of the Earth, which produce to *Z*, draw *AB* perpendicular to *CA* and let it repreſent the denſity of the air at the ſurface, and *PM* repreſent the denſity at any altitude *AP*, and let



BMD be a curve paſſing through the extremities of all the ordinates *PM*. Then (Art. 112) as *AP* increaſes in arithmetic progreſſion, the denſity *PM* will decreaſe in geometric progreſſion; hence *PM* is the number correſponding to the logarithm AP, and BD is called the logarithmic curve, whoſe ſubtangent PT is the modulus of the ſyſtem. Let AQ be the altitude of an homogeneous * atmoſphere whoſe denſity is AB, and complete the parallelogram $BAQR$. Now conſider the whole atmoſphere AZ, and the homogeneous atmoſphere AQ to be divided into an indefinite number of laminæ of equal thickneſs; then (Art. 112) the whole preſſure on AB in both caſes may be meaſured by the ſum of all the denſities of theſe laminæ, and the denſity being as the lamina PM and AB reſpectively, the preſſures will be as the ſum of all the PM's, and the ſum of all the AB's, or as (Fluxions, Art. 49, Ex. 4.) $AB \times PT$, and $AB \times AQ$; but theſe preſſures are equal; hence $AB \times PT = AB \times AQ$, conſequently $AQ = PT$; the modulus of this ſyſtem of logarithms is therefore the altitude of an homogeneous atmoſphere.

For the general inveſtigation of the denſity of the air, when the force of gravity is ſuppoſed to vary as any power of the diſtance from the Earth's center, ſee the treatiſe on Fluxions, page 216.

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* An homogeneous atmoſphere is an atmoſphere ſuppoſed to be of the ſame weight as that which ſurrounds the Earth, and whoſe denſity is uniform and equal to the denſity of the air at the Earth's ſurface.

* * *

SECTION VII.

ON THE BAROMETER.

Prop. LIII.

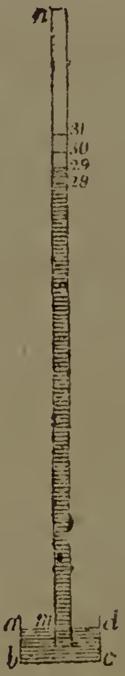
To make a Barometer.

115. If a glaſs tube above 31 inches long, hermetically ſealed at one end, be filled with mercury, and then it's open end be immerſed in a baſon of the ſame fluid, the altitude at which the mercury will ſtand in the tube above the ſurface of the mercury in the baſon is between 28 and 31 inches. A tube thus filled is called a *Barometer.*

Prop. LIV.

The mercury is ſuſpended in the tube of a barometer by the preſſure of the air upon the ſurface of the mercury in the baſon.

116. For if a barometer *mn* be put under the receiver of an air pump, and the air be exhauſted, as you continue to exhauſt the air, and conſequently to diminiſh it's preſſure upon the ſurface of the mercury in the baſon, the mercury in the tube will continue to deſcend, and when no ſenſible quantity of air is left, the altitude of the mercury will not be ſenſibly above that in the baſon; and upon admitting the air again into the receiver, the mercury will riſe in the tube to it's former height.



As the denſity of the air, and conſequently it's compreſſing force, is ſubject to a variation, the altitude of the mercury muſt be ſubject alſo to a correſponding variation; it is always however contained between the limits of 28 and 31 inches. A tube thus filled is therefore graduated from 28 to 31 inches.

117. GALILEO was the firſt perſon who diſcovered the preſſure of the air. He found by experiment, that water could be raiſed, by the common pump, to a certain height, and no higher; whereas, had nature abhorred a vacuum, according to the opinion of ſome of the philoſophers at that time, the water might have been raiſed to any height. He conjectured therefore, that it was owing to the air's gravitation; the truth of which was afterwards confirmed by his pupil TORRICELLIUS, who conſidered, that if the preſſure of the air could ſupport a column of water 35 feet high, which is about the mean height to which a pump can raiſe water, it could ſuſpend a column of mercury, whoſe denſity is about 14 times as great, only about one 14th part of 35 feet high, or about 30 inches; he accordingly tried the experiment, and found that the mercury ſtood at the altitude which he expected. Thus he clearly proved the gravitation of the air; and hence this is called the *Torricellian* experiment; and the vacuum which is left above, when the mercury deſcends from the top of the tube after immerſing it in the baſon, is called the *Torricellian* vacuum. When the tube is filled with great care, this vacuum is ſuppoſed to be the moſt perfect that can be made.

PROP. LV.

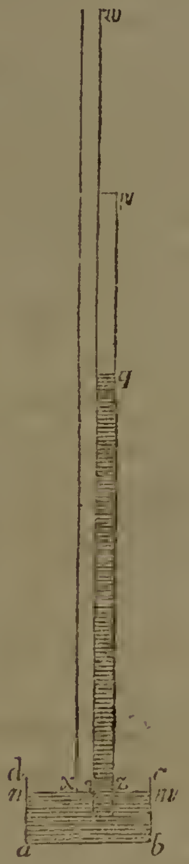
To find the height of an homogeneous atmoſphere.

118. The mercury in the tube of a barometer is (Art. 116) ſuſtained by the preſſure of the air; and (Art. 31) when two fluids communicate, the altitudes at which they ſtand are inverſely as their ſpecific gravities. Let us take the ſpecific gravity of air to that of water as 1 : 850, the barometer ſtanding at 30 inches. Now if we take the ſpecific gravity of water to mercury as 1 : 14, we ſhall have the ſpecific gravity of air to that of mercury as 1 : 12390; hence, (Art. 31) 1 : 12390 :: 30 in. : 12390 × 30 = 371700 in. = 5,63 miles, the height of an homogeneous atmoſphere, or an atmoſphere of the ſame weight as the preſent atmoſphere, and whoſe ſpecific gravity is every where the ſame as that of the air at the earth's ſurface. If we take the ſpecific gravities of air and water as 1 : 885, when the barometer ſtands at 29½ inches, we ſhall have the altitude of an homogeneous atmoſphere 5,77 miles. The ſpecific gravity of mercury has been here ſuppoſed 14, that of water being 1; but when the mercury is very pure, it's ſpecific gravity has been found to be only 13,6. To determine with accuracy the height of an homogeneous atmoſphere by this method, the ſpecific gravity of the mercury in the baromometer, at the time of obſervation, ſhould be determined, as it is ſubject to a ſmall variation from the different temperatures of the air.

PROP.

Prop. LVI.

The weight of the mercury in the barometer (the tube being cylindrical) above the level of that in the baſon, is equal to the weight of a cylinder of air of the ſame baſe reaching to the top of the atmoſphere.



119. Let qz be the altitude of the mercury in the tube pv; take a cylindrical column xvw of the air, whoſe baſe xv is equal to vz that of the mercury. Now the ſection $nxvm$ of mercury being at reſt, every point thereof muſt be equally preſſed, and therefore equal parts muſt be equally preſſed; but the preſſures

on xv, vz ariſe from the incumbent columns xw of air and vq of mercury, and theſe being perpendicular cylindrical columns the preſſures are equal to their weights; conſequently the weights of theſe columns are equal.

Some have found it difficult to conceive why the weight of the mercury in the tube ſhould not be equal to the weight of the air preſſing upon the *whole* ſurface of the mercury in the baſon. This difficulty has ariſen from their not making a proper diſtinction between preſſure and weight; the column of mercury gives a preſſure upwards to the ſurface of the mercury in the baſon equal to the weight of the whole incumbent air, but as fluids preſs equally in all directions, this preſſure which the mercury gives is as much greater than it's weight as the ſurface of mercury in the baſon is greater than the orifice of the tube. It is a fact ſimilar to the hydroſtatical paradox, where a ſmaller weight ſuſtains a greater.

Prop. LVII.

When the mercury in the barometer ſtands at 30 *inches, the preſſure of the air upon every ſquare inch is about* 15*lb. avoirdupoiſe.*

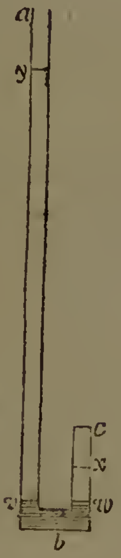
120. For by the laſt Article, a column of air, of mean denſity, whoſe baſe is 1 ſquare inch, preſſes as much as a column of mercury of the ſame baſe 30 inches high, the weight of which is about 15lb. avoirdupoiſe.

121. Cor. Hence, if we take the ſurface of a middle ſize man to be 14½ ſquare feet, when the air is lighteſt

it's preſſure on him is 13,2 tons; and when heavieſt, it is 14,3, the difference of which is 2464 lb. This difference of preſſures muſt greatly affect us in regard to our animal functions, and conſequently in reſpect to our health, more eſpecially when the change takes place in a ſhort time. The preſſure of the air upon the whole ſurface of the Earth is about 7767029797356 3429 tons.

PROP. LVIII.

The denſity of the air is in proportion to the force which compreſſes it.



122. Let *abc* be a glaſs cylindrical tube, hermetically ſealed at *c*, and let the bottom be covered with mercury, whilſt the air in *wc* is in it's natural ſtate. Pour in mercury at *a* and it will force the mercury to riſe in *wc*, and continue to pour in, till the mercury ſtands at *y* as high above the point

x to which it has now riſen in wc, as the altitude of the mercury in the common barometer; then that column of mercury (Art. 119) is equivalent to the weight of the column of air incumbent upon it, hence the preſſure againſt the air in cx is now twice as much as it was againſt the air in cw, and cx is obſerved to be $=\frac{1}{2}cw$; hence the air being compreſſed into half the ſpace, the denſity is doubled. In like manner, if another column of mercury of the ſame altitude be added, cx is found to be $=\frac{1}{3}cw$; thus the compreſſing force is made three times greater, and the denſity is three times greater. In this manner, the compreſſing force is found in any other caſe to be in proportion to the denſity. The ſame is obſerved to be true in all kinds of factitious airs, upon which experiments have been made.

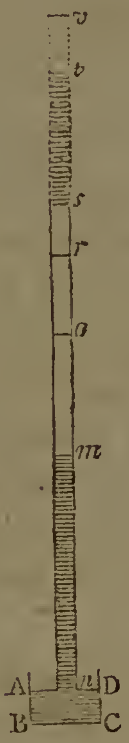
123. By increaſing the compreſſing force of the air, the particles are brought nearer together, but are kept from coming into contact by their repulſive force; theſe forces muſt therefore be equal, when the fluid is at reſt. The repulſive force is what we uſually call the air's elaſticity; hence the elaſticity of the air being in proportion to it's compreſſive force, muſt be alſo in proportion to it's denſity.

Prop. LIX.

If the tube ns of a barometer be perfectly cylindrical, and be in part only filled with mercury, and then it's open end be immerſed in a baſon of the ſame fluid, the mercury will ſink below the point, called the ſtandard altitude, or the point at which it would have ſtood if no air had been

been left in; and the ſtandard altitude will be to the depreſſion below that altitude as the ſpace occupied by the air after the immerſion to the ſpace occupied before.

124. Let *rs* be equal to the ſpace occupied by the air before the tube is immerſed, or when the air is in it's natural ſtate; after the immerſion of the tube into the baſon *ABCD* let the mercury ſink to *m*; then the air which, in it's natural ſtate, occupied the ſpace *rs*, now occupies the ſpace *ms*, and the ſpace occupied by the ſame quantity being inverſely as the denſity, or (Art. 123) inverſely as the elaſticity, we have the elaſticity in *rs* : the elaſticity in *sm* :: *sm* : *rs*. Let *no* be the height at which the mercury would



would have ſtood if no air had been left in the tube, or the height of the mercury in the barometer. Then (Art. 122) the compreſſive force of the air, and (Art. 123) conſequently it's elaſticity, when it occupied the ſpace *rs*, would ſupport a column of mercury *no*, becauſe the air, when it occupied that ſpace, was in it's natural ſtate; and the elaſticity of the air when it occupies the ſpace *sm* would ſupport a column of mercury *mo*, becauſe it depreſſes the mercury from *o* to *m*; hence the elaſticity of the air in *rs* : the elaſticity in *sm* :: *on* : *om*; conſequently *on* : *om* :: *sm* : *sr*.

This propoſition may be applied to the ſolution of two problems; for we may either give the quantity of air left in before immerſion, to find the altitude of the mercury after; or we may give the altitude of the mercury after immerſion, to find the quantity of air left in before. As the ſtandard altitude *no* (Art. 116) is ſubject to a variation, it has been uſual in this caſe to aſſume it 30 inches; but when accuracy is required, it muſt be taken equal to the height of the mercury in the barometer at the time.

Ex. 1. Let the length *ns* of the tube be 35 inches, and the depreſſion *om* below the ſtandard altitude *no* (=30 in.) be 10 inches, to find the quantity of air left in before inverſion.

As $ns=35$, and $no=30$, we have $so=5$; alſo $om=10$; hence $sm=15$; therefore, $30 : 10 :: 15 : rs = \frac{10\times15}{30} = 5$ inches.

Ex. 2. Let 5 inches of air be left in the ſame tube before inverſion, to find the altitude of the mercury after.

In this caſe the point m being unknown, the ſecond and third terms of the proportion are unknown; put therefore $x=om$, then $x+5=sm$; hence $30 : x :: x+5 : 5$, therefore $x^2+5x=150$, conſequently $x=10$ or -15. The anſwer $+10$ ſhows that the mercury will ſtand at 10 inches below o; and the anſwer -15 ſhows, that if the tube were continued to v, and ot taken equal to 15 inches, and the ſpace st were filled with mercury, the ſpace tv above being a vacuum, that this column st of mercury would alſo be ſupported by the elaſticity of the air in sm. In fact, $st=om$, and therefore the elaſticity of the air which depreſſes a column om muſt neceſſarily ſuſtain an equal column st.

The experiments agreeing with the concluſions here deduced, it follows that the compreſſive force of the air is as it's denſity, that being the principle upon which the demonſtration is founded.

PROP. LX.

If a be the altitude of the mercury in a barometer at the bottom of an hill, and b the altitude at the top, the altitude of the hill will be $11,626 \times \log. \frac{a}{b}$ *miles.*

125. For the weight of mercury ſuſpended by the preſſure of the air at the bottom : the weight ſuſpended at the top $:: a : b :: 1 : \frac{b}{a}$; hence the compreſſive

forces of the air at theſe two points, by ſuſtaining theſe columns of mercury, muſt be as $1 : \frac{b}{a}$; but the compreſſive forces are as (Art. 122) the denſities; therefore the denſities below and above are as $1 : \frac{b}{a}$, conſequently the denſity above will be $\frac{b}{a}$, that at the ſurface being unity. Hence the rarity will be $\frac{a}{b}$; ſubſtitute this for m (Art. 113.) and the altitude $x = 11{,}626 \times \log. \frac{a}{b}$. The difference of the temperatures of the air at the bottom and top is not here conſidered, which will make a ſmall alteration. This correction may be applied, by obſerving with two thermometers the temperatures at each point, and then allowing for that difference; but the inveſtigation of this falls not within our preſent plan.

Dr. HALLEY's *Account of the Riſing and Falling of the Mercury in a Barometer, upon the Change of Weather.*

126. To account for the different heights of the mercury at ſeveral times, it will not be unneceſſary to enumerate ſome of the principal obſervations made upon the barometer.

1ſt. The firſt is, that in calm weather, when the air is inclined to rain, the mercury is commonly low.

2ly. That in ſerene, good, ſettled weather, the mercury is generally high.

3ly. That upon very great winds, though they be not accompanied with rain, the mercury ſinks loweſt of all, with relation to the point of the compaſs the wind blows upon.

4ly. That *cæteris paribus*, the greateſt heights of the mercury are found upon eaſterly and north-eaſterly winds.

5ly. That in calm froſty weather, the mercury generally ſtands high.

6ly. That after very great ſtorms of wind, when the quickſilver has been low, it generally riſes again very faſt.

7ly. That the more northerly places have greater alterations of the Barometer than the more ſoutherly.

8ly. That within the tropicks and near them, thoſe accounts we have had from others, and my own obſervations at St. Helena, make very little or no variation of the height of the mercury in all weathers.

Hence I conceive, that the principal cauſe of the riſe and fall of the mercury, is from the variable winds which are found in the temperate zones, and whoſe great inconſtancy here in England is moſt notorious.

A ſecond cauſe is the uncertain exhalation and precipitation of the vapours lodging in the air, whereby it comes to be at one time much more crouded than at another, and conſequently heavier; but this latter in a great meaſure depends upon the former. Now from theſe principles I ſhall endeavour to explicate the ſeveral phænomena of the barometer, taking them in the ſame order I laid them down.

1ſt. The mercury's being low inclines it to rain, becauſe the air being light, the vapours are no longer ſupported thereby, being become ſpecifically heavier than the medium wherein they floated; ſo that they deſcend towards the Earth, and in their fall meeting with other aqueous particles, they incorporate together and form little drops of rain. But the mercury's being at one time lower than at another, is the effect of two contrary winds blowing from the place where the barometer ſtands; whereby the air of that place is carried both ways from it, and conſequently the incumbent cylinder of air is diminiſhed, and accordingly the mercury ſinks. As for inſtance, if in the German ocean it ſhould blow a gale of weſterly wind, and at the ſame time an eaſterly wind in the Iriſh ſea, or if in France it ſhould blow a northerly wind, and in Scotland a ſoutherly, it muſt be granted me that, that part of the atmoſphere impendent over England would thereby be exhauſted and attenuated, and the mercury would ſubſide, and the vapours which before floated in thoſe parts of the air of equal gravity with themſelves, would ſink to the Earth.

2ly. The greater height of the barometer is occaſioned by two contrary winds blowing towards the place of obſervation, whereby the air of other places is brought thither and accumulated; ſo that the incumbent cylinder of air being increaſed both in height and weight, the mercury preſſed thereby muſt needs riſe and ſtand high, as long as the winds continue ſo to blow; and then the air being ſpecifically heavier, the vapours are better kept ſuſpended, ſo that they have no inclination to precipitate and fall down in drops; which is the reaſon of the ſerene good weather, which attends the greater heights of the mercury.

3ly. The mercury ſinks the loweſt of all by the very rapid motion of the air in ſtorms of wind. For the tract or region of the Earth's ſurface, wherein theſe winds rage, not extending all round the globe, that ſtagnant air which is left behind, as likewiſe that on the ſides, cannot come in ſo faſt as to ſupply the evacuation made by ſo ſwift a current; ſo that the air muſt neceſſarily be attenuated when and where the ſaid winds continue to blow, and that more or leſs according to their violence; add to which, that the horizontal motion of the air being ſo quick as it is, may in all probability take off ſome part of the perpendicular preſſure thereof: and the great agitation of its particles is the reaſon why the vapours are diſſipated, and do not condenſe into drops ſo as to form rain, otherwiſe the natural conſequence of the air's rarefaction.

4ly. The mercury ſtands the higheſt upon an eaſterly or north-eaſterly wind, becauſe in the great Atlantick ocean, on this ſide the 35th degree of north latitude, the weſterly and ſouth-weſterly winds blow almoſt always Trade, ſo that whenever here the wind comes up at eaſt and north-eaſt, it is ſure to be checked by a contrary gale as ſoon as it reaches the ocean; wherefore according to what is made out in our ſecond remark, the air muſt needs be heaped over this iſland, and conſequently the mercury muſt ſtand high, as often as theſe winds blow. This holds true in this country, but is not a general rule for others where the winds are under different circumſtances; and I have ſometimes ſeen the mercury here as low as 29 inches upon an eaſterly wind, but then it blew exceeding hard, and ſo comes to be accounted for by what was obſerved upon the third remark.

5ly. In calm froſty weather the mercury generally ſtands high, becauſe (as I conceive) it ſeldom freezes but when the winds come out of the northern and north-eaſtern quarters, or at leaſt unleſs thoſe winds blow at no great diſtance off; for the northern parts of Germany, Denmark, Sweden, Norway, and all that tract from whence north-eaſtern winds come, are ſubject to almoſt continual froſt all the winter; and thereby the lower air is very much condenſed, and in that ſtate is brought hitherwards by thoſe winds, and being accumulated by the oppoſition of the weſterly wind blowing in the ocean, the mercury muſt needs be preſt to a more than ordinary height; and as a concurring cauſe, the ſhrinking of the lower parts of the air into leſſer room by cold, muſt needs cauſe a deſcent of the upper parts of the atmoſphere to reduce the cavity made by this contraction to an *æquilibrium.*

6ly. After great ſtorms of wind, when the mercury has been very low, it generally riſes again very faſt. I once obſerved it to riſe $1\frac{1}{2}$ inch in leſs than 6 hours after a long continued ſtorm of ſouth-weſt wind. The reaſon is, becauſe the air being very much rarefied, by the great evacuations which ſuch continued ſtorms make thereof, the neighbouring air runs in the more ſwiftly to bring it to an *æquilibrium*; as we ſee water runs the faſter for having a great declivity.

7ly. The variations are greater in the more northerly places, as at Stockholm greater than at Paris (compared by Mr. PASCALL*) becauſe the more northerly parts have uſually greater ſtorms of wind than the more ſoutherly, whereby the mercury ſhould ſink lower in that extreme; and then the northerly winds bringing the condenſed and ponderous air from the neighbourhood of the pole, and that again being checked by a ſoutherly wind at no great diſtance, and ſo heaped, muſt of neceſſity make the mercury in ſuch caſe ſtand higher in the other extreme.

8ly. Laſtly, this remark, that there is little or no variation near the equinoctial, as at Barbadoes and St. Helena, does above all others confirm the hypotheſis of the variable winds being the cauſe of theſe variations of the height of the mercury; for in the places above named there is always an eaſy gale of wind blowing nearly upon the ſame point, viz. E.N.E. at Barbadoes, and E.S.E. at St. Helena, ſo that there being no contrary currents of the air to exhauſt or accumulate it, the atmoſphere continues much in the ſame ſtate: however upon hurricanes (the moſt violent of ſtorms) the mercury has been obſerved very low, but this is but once in two or three years, and it ſoon recovers its ſettled ſtate of about 29½ inches.

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* Equilibre des Liqueurs.

. The principal objection againſt this doctrine is that I ſuppoſe the air ſometimes to move from thoſe parts where it is already evacuated below the *æquilibrium*, and ſometimes again towards thoſe parts where it is condenſed and crouded above the mean ſtate, which may be thought contradictory to the laws of ſtaticks, and the rules of the *æquilibrium* of fluids. But thoſe that ſhall conſider how when once an impetus is given to a fluid body, it is capable of mounting above its level, and checking others that have a contrary tendency to deſcend by their own gravity will no longer regard this as a material obſtacle; but will rather conclude, that the great analogy there is between the riſing and falling of the water upon the flux and reflux of the ſea, and this of accumulating and extenuating the air, is a great argument for the truth of this hypotheſis. For as the ſea, over againſt the coaſt of Eſſex, riſes and ſwells by the meeting of the two contrary tides of flood, whereof the one comes from the S. W. along the channel of England, and the other from the north, and on the contrary ſinks below its level upon the retreat of the water both ways, in the tide of ebb; ſo it is very probable, that the air may ebb and flow after the ſame manner; but by reaſon of the diverſity of cauſes whereby the air may be ſet in moving, the times of theſe fluxes and refluxes thereof are purely caſual, and not reducible to any rule, as are the motions of the ſea, depending wholly upon the regular courſe of the moon.

SECTION VIII.

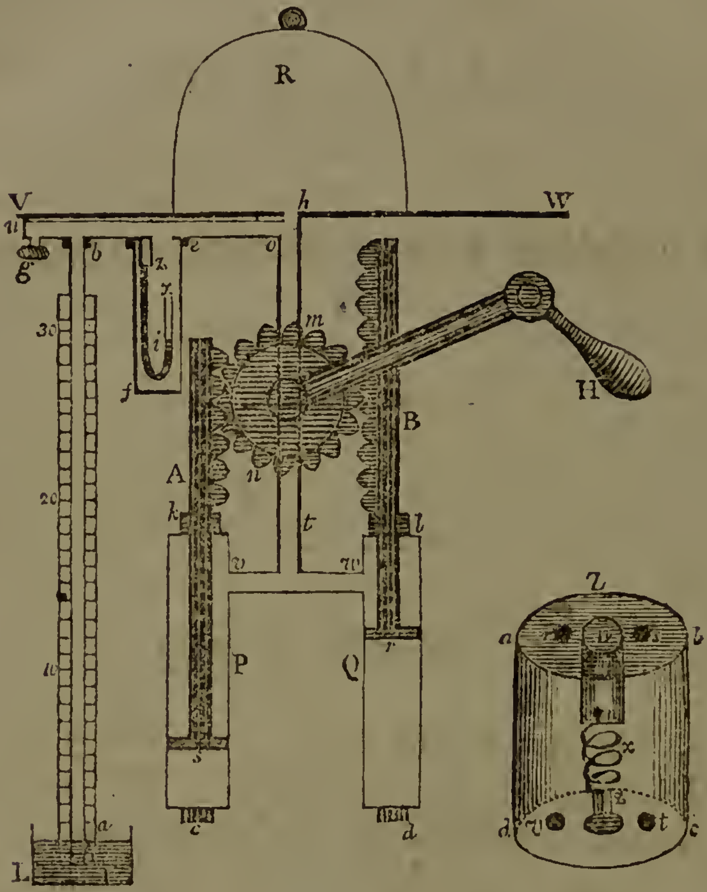
ON THE AIR PUMP, AND CONDENSER.

Prop. LXI.

To conſtruct an Air Pump.

127. The Air Pump which I uſe in my experimental Lectures is thus conſtucted. WV is a braſs plate ground perfectly plane, and ſtrengthened on the under ſide with ribs; at h there is a ſmall orifice, over which ſtands a glaſs veſſel R, called a receiver, the edge of which is alſo ground truly plane, ſo that if a little greaſe be put upon the edge before it is placed on the receiver, it will be air-tight; in general however a piece of leather well prepared with greaſe is laid upon the plate for the receiver to ſtand upon; but you may make a more perfect exhauſtion by the other method, on account of the air which the leather will give out; in this leather there is a hole made correſponding to h in the plate. From h a braſs pipe ht

deſcends, and turning each way at the bottom enters the two barrels, *P*, *Q*, at *v* and *w*. At the bottom of



each barrel there is a ſmall hole againſt which there are two pieces *c*, *d* ſcrewed on, containing the valves, each of which is repreſented by the figure *Z*, which is a ſolid piece *abcd* of braſs, through the middle of which there is a cylindrical hole, partly filled with a ſolid braſs cylinder, againſt the bottom *m* of which a ſpiral ſpring *x* acts, which reſts below againſt a ſcrew *z*, by means of which the ſpring may be rendered ſtronger or weaker; through the braſs there are alſo two holes, one from *r* to *v*, and the other from *s* to *t*; and over the top *ab* there is tied a piece of oiled ſilk, having two holes correſponding to *r* and *s*; and when this piece *Z* is ſcrewed on to the bottom of the barrel, the end *n* of the cylinder *nm* is preſſed againſt the hole in the barrel, by means of the ſpring *x*. The barrels are truly cylindrical, having each a ſucker *r*, *s*, (without a valve) ſurrounded with leather, and fitted ſo cloſe to the barrel as to be air-tight; theſe ſuckers are fixed to two braſs rods *A*, *B*, having cogs above; *mn* is a ſmall wheel with cogs acting on thoſe of the rods, and moved by an handle *H*, which being turned backwards and forwards, the rods *A*,*B* and conſequently the ſuckers *s*,*r* aſcend and deſcend alternately. From the top of the pipe *ht* there proceeds another pipe *ou*, into an orifice of which there is fixed a glaſs tube *ab*, having it's lower end immerſed in a baſon *L* of quickſilver; this tube is called the *gage*, at the back of which there is fixed a frame of wood, which is graduated from the mercury in the baſon up to 31 inches. At *g* there is a ſcrew, by unſcrewing which you can admit the air into the pipe *ou* when it is exhauſted. The rods *A*, *B*, paſs each through a collar of leathers at *k* and *l*, which are air-tight. The ſupporters to the whole of this are here omitted, as they would have rendered the figure confuſed, and have been of no uſe for the underſtanding of the inſtrument. This being the conſtruction, the exhauſtion takes place in the following manner.

128. Turn the handle, and bring the ſucker *r* down to the bottom of the barrel, then the ſucker *s* will

be carried juſt above the orifice v; and by turning the handle in the contrary direction, s will be depreſſed to the bottom, and r will riſe juſt above the orifice w. Now upon the deſcent of s, it muſt manifeſtly force all the air in the barrel P before it, the ſides being air-tight; the air therefore will depreſs the cylinder nm (Fig. Z) and eſcape through the holes rv, st; after which the ſcrew x will force mn up againſt the orifice at the bottom of the barrel, and prevent any air from returning into it. Then elevate s and depreſs r, and r in like manner will force out all the air before it. Now as s aſcends, it leaves a vacuum between s and the bottom; but when s has gotten above v, the air will ruſh from the pipe t, which communicates with the receiver R and gage ab, into this vacuum, the conſequence of which is, that the air in the receiver and gage becomes rarified by being expanded into a greater ſpace; and as this muſt take place every time each ſucker deſcends, or at each turn of the handle, there muſt be a continued exhauſtion, and conſequently a continued rarefaction of the air in the receiver and gage. But beſides this gage, there is another included in a glaſs cylinder ef which has alſo a communication with the pipe ou; in this there is a bent glaſs tube zix, hermetically ſealed at the upper end z, and filled with mercury to i, as repreſented by the ſhaded part. Then when the air is exhauſted to a conſiderable degree, the preſſure of the air upon the mercury at i will not be able to ſuſtain the mercury in the other leg, and therefore it will deſcend, and the two ſurfaces will approach to the ſame level, and if you could make a perfect exhauſtion, they would ſtand in the ſame horizontal line; the difference of the altitudes therefore (meaſured upon a ſcale which lies againſt them) ſhows how much there wants of a perfect vacuum. If to the height of the mercury in the other gage, you add the difference of the altitudes in this gage, it gives the altitude in the gage ab if you could make a perfect vacuum, or it gives the altitude at which the Barometer ſtands at that time. By this method you may try whether a Barometer be properly filled and graduated.

LEMMA.

129. Let a quantity a be diminiſhed till it becomes ſucceſſively b, c, d, &c. and let the decrements $a-b$, $b-c$, $c-d$, &c. be always in proportion to the quantities themſelves a, b, c, d, &c. then will both theſe quantities and their decrements be in geometrical progreſſion.

For by ſuppoſition, $a : a-b :: b : b-c :: c : c-d ::$ &c. hence dividendo, $a : b :: b : c :: c : d ::$ &c. Alſo alternando, $a : b :: a-b : b-c$, $b : c :: b-c : c-d$, &c. hence $a-b : b-c :: b-c : c-d ::$ &c.

Prop. LXII.

If b repreſent the capacity of one of the barrels, and r that of the receiver, together with the pipes and gages connected with it; then the quantity of air extracted after every turn : the quantity before that turn $:: b : 2b+r$; and the quantity left in : the quantity before $:: b+r : 2b+r$.

130. For

130. For conceive the ſucker r to be down and s to be up, and the receiver, pipes, gages and barrels, which all now communicate, to be filled with air; then as the whole capacity of theſe is $2b+r$, the quantity of air may be repreſented by $2b+r$, from which, by the deſcent of s, the quantity b will be driven out; and this muſt evidently be the caſe at every turn. And as the quantity b is taken away from $2b+r$, there muſt remain the quantity $b+r$.

Cor. Hence the quantity taken away at every turn being always in the ſame ratio to the whole quantity before the turn, the air can never be all exhauſted.

Prop. LXIII.

The denſity of the air in the receiver at firſt : the denſity after t turns :: $\overline{2b+r}|^t : \overline{b+r}|^t$.

131. For the denſity is (Art. 4.) as the quantity of air contained in the ſame ſpace. Now the quantity before any turn : the quantity after :: $2b+r : b+r$ by Art. 130. and therefore the denſity at every turn is diminiſhed in the ſame ratio; hence, by the compoſition of ratios, after t turns, the denſity is diminiſhed in the ratio of $\overline{2b+r}|^t : \overline{b+r}|^t$.

Hence the denſity is diminiſhed in geometrical progreſſion.

Prop. XLIV.

When the denſity of the air is diminiſhed in the ratio of $n : 1$, *the number of turns* $t = \dfrac{\log.\ n}{\log.\ \overline{2b+r} - \log.\ \overline{b+r}}$.

132. For (Art. 131) $n : 1 :: \overline{2b+r}|^{t} : \overline{b+r}|^{t}$, hence $n = \frac{\overline{2b+r}|^{t}}{\overline{b+r}|^{t}}$, conſequently (Fluxions Art. 109) log. $n = t \times \log. \frac{2b+r}{b+r} = t \times \overline{\log. \overline{2b+r} - \log. \overline{b+r}}$; hence $t = \frac{\log. n}{\log. \overline{2b+r} - \log. \overline{b+r}}$.

PROP. LXV.

As the air is exhauſted, the mercury will riſe in the gage; and the defects *of the mercury in the gage from the ſtandard altitude, after each ſucceſſive turn, form a geometric ſeries, the ratio of whoſe terms is* $2b+r : b+r$.

133. For as the denſity of the air within the gage, and conſequently (Art. 122) it's compreſſing force on the mercury, is diminiſhed at every turn, the compreſſing force of the air upon the mercury in the baſon, which remains the ſame, muſt cauſe the mercury to riſe in the gage. If all the air were exhauſted, the mercury would riſe as high as in the common Barometer, or ſo what is called the ſtandard altitude. Now the compreſſing force of the quantity of air left in, prevents the mercury from riſing to the ſtandard altitude, and therefore it's compreſſing force muſt be equivalent to a column of mercury equal to the defect; therefore the defect, being as the compreſſing force, muſt be (Art. 122) in proportion to the denſity, which, at every turn, diminiſhes in the ratio of $2b+r : b+r$, by Art. 131.

PROP.

Prop. LXVI.

The aſcents *of the mercury in the gage, at each ſucceſſive turn, form a geometric ſeries, the ratio of whoſe terms is* $2b+r : b+r$.

134. The defects of the mercury from the ſtandard altitude diminiſh in the ratio of $2b+r : b+r$, and the differences of theſe defects are the ſucceſſive aſcents of the mercury; but, by the Lemma, if a ſet of quantities decreaſe in geometrical progreſſion, their differences will alſo decreaſe in the ſame geometrical progreſſion; hence the aſcents of the mercury ſucceſſively decreaſe in the ratio of $2b+r : b+r$.

135. The various properties of the air are very readily ſhown by the air pump; as in the following experiments:

Ex. 1. Air is neceſſary for the *production* of ſound.

For if a bell be put under the receiver of an Air Pump, and the air be exhauſted, the bell, when ſtruck, cannot be heard; and if the air be gradually let in the ſound will gradually increaſe.

Ex. 2. Air is neceſſary for the *propagation* of ſound.

For if a receiver be put over a bell, and then another receiver over that, and the air be exhauſted from between them, no ſound is heared; the ſound therefore is not propagated through the vacuum.

Ex. 3. Air is neceſſary for the exiſtence of fire.

For if a candle be put under the receiver and the air be exhauſted, it immediately goes out.

Ex. 4. Air is neceſſary for the exiſtence of animal life.

For moſt animals put under the receiver die almoſt immediately upon exhauſting the air, and probably all would, could we make a perfect vacuum.

Ex. 5. The preſſure of the air is rendered viſible, by taking away the air from one ſide of a body, whilſt it continues on the other.

For if a bladder be tied over the top of a glaſs receiver, and the air be exhauſted from within, at every exhauſtion, the preſſure of the air upon the bladder will continue to depreſs it, until it burſts with a very great exploſion.

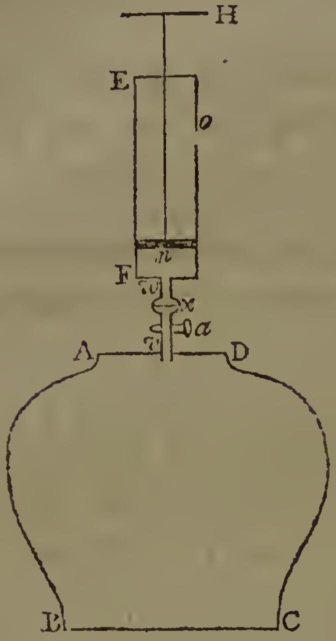
Theſe are a few of the properties of the air which are ſhown by this inſtrument; but the experiments are too many to be all here enumerated.

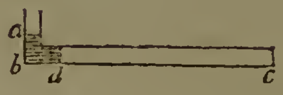
Prop. LXVII.

To conſtruct a Condenſer.

136. A Condenser is thus conſtructed. *ABCD* is a ſtrong veſſel called a receiver, made either of glaſs or metal; if of glaſs, upon the top there is laid a braſs plate with a ſtop cock *a*, having under it a prepared piece of leather to make it air-tight, and alſo a like plate at the bottom. Into the cock at *x* there

is ſcrewed a ſyringe *E F*, having a ſucker *n*, which is moved by a handle at *H*; at *w* there is a





valve which opens downwards, and at *o* there is an orifice. Now let the ſucker be drawn up above the orifice *o*, and both the barrel of the ſyringe and the receiver to be filled with air in it's natural ſtate. Then upon forcing down the ſucker, the air opens the valve at *w*, and a barrel of common air is forced into the receiver. Upon raiſing again the ſucker *n* a vacuum is left under it, the valve preventing the air from returning; and when the ſucker gets above *o*, the air will immediately ruſh in and fill the barrel; thus upon every deſcent of the ſucker you force into the receiver a barrel of common air, and conſequently you condenſe the air in the receiver.

After the receiver is charged, the ſtop cock at *a* may be turned to prevent the return of the air, and the ſyringe may be taken off, and any other apparatus may be ſcrewed on for experiments with the condenſed air in the receiver.

PROP. LXVIII.

If b repreſent the capacity of the barrel of the ſyringe, and r *that of the receiver, then after* t *deſcents of the ſucker, the denſity of the air in the receiver will be to the denſity at firſt in the ratio of* $r+tb : r$.

137. For the quantity of air at firſt may be repreſented by r, and after t deſcents of the ſucker, a quantity repreſented by tb will be forced into the receiver, and therefore the whole quantity in it will be $r+tb$; hence, (Art. 4) the denſity after t deſcents : the denſity at firſt :: $r+tb : r$.

Cor. Hence the denſities after any number of ſucceſſive deſcents are in arithmetic progreſſion.

138. If *abc* be a glaſs tube with the end at *a* open, and the other end hermetically ſealed, and a ſmall quantity of mercury put in ſo as to leave the air in *dc* in it's natural ſtate; then if this be put into the receiver with the part *bc* horizontal, and the air be condenſed, the condenſed air preſſing on the mercury will force it towards *c*, and the air in *dc* will continue of the ſame denſity as that in the receiver. Now as the denſity is inverſely as the ſpace occupied by the ſame quantity (Art. 36.) the denſity in *dc*, and conſequently in the receiver, is inverſely as *dc*; when therefore *dc* is diminiſhed until it be *n* times leſs

than it was at firſt, the denſity will be increaſed *n* times. Hence, as the denſity, after any number of ſucceſſive turns, increaſes in arithmetic progreſſion, the reciprocal of the ſpaces will be in arithmetic progreſſion, and therefore the ſpaces themſelves will decreaſe in muſical progreſſion. This inſtument is called a *gage*.

139. A bell in condenſed air ſounds louder than in air in it's natural ſtate. Fire Engines, Air Guns, Artificial Fountains, ſome kinds of Forcing Pumps, &c. act by condenſed air.

* * *

SECTION IX.

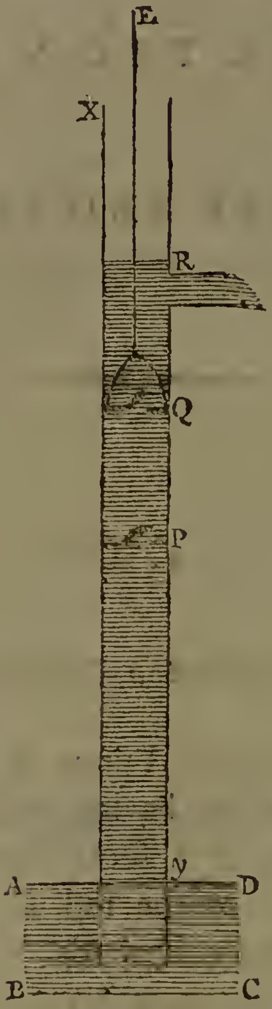
ON PUMPS AND SYPHONS.

PROP. LXIX.

To conſtruct a Common Pump.

140. The common Pump is thus conſtructed. *Xy* is a hollow cylinder having its lower end in water, *P* is a fixed ſucker, *Q* a ſucker moveable by means of an handle fixed to the rod *E*, and each ſucker has a valve opening upwards. Now let us ſuppoſe *Q* to deſcend as low as it can, and each valve to be ſhut, and that the pump has at preſent no water in it; then when *Q* aſcends, the air between *P* and *Q* will follow it, and conſequently it will become rarified, therefore the air under *P* being now denſer than the air above, it will open the valve at *P* and ruſh into *PQ*, and the whole air within being thus rarified, it will not open the valve at *Q*, which is preſſed

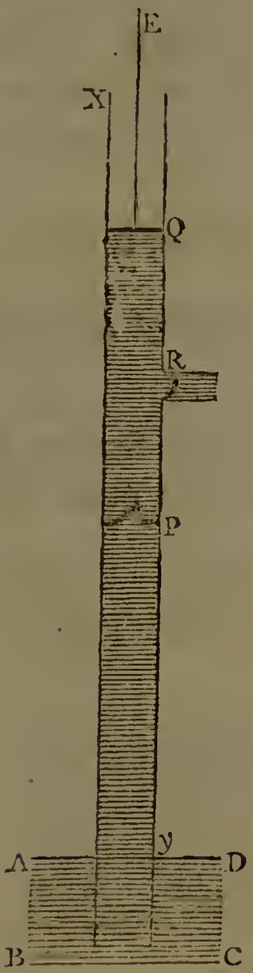
down with air that is not rarified. The air therefore in the pump being rarified, the preſſure of the air upon



the ſurface of the water without the pump will force the water a little way up the pump. Now when *Q* deſcends, it will preſs down the air under it, and that air will ſhut the valve at *P* by preſſing *upon* it, but it will open the valve at *Q* by preſſing *under* it, and thus ſome of the air will eſcape. Then when *Q* aſcends again, the preſſure of the air upon it's valve will ſhut it, and the ſame operation will be repeated. Thus at each deſcent of *Q* the water will riſe, till at length it comes up to *Q*, and then upon the deſcent of *Q* it will open it's valve and get above the ſucker, and the ſucker then being drawn up, it will carry the water up and throw it out of the ſpout *R*.

Prop. LXX.

To conſtruct a forcing Pump.



141. Here the ſucker *Q* has no valve, and the air

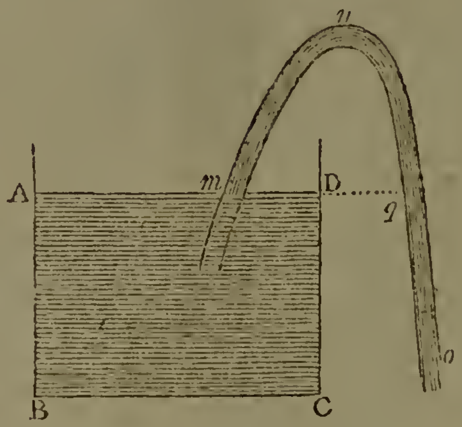
between *P* and *Q* is, by depreſſing the ſucker *Q*, expelled through a valve opening outwards at *R*, inſtead of being expelled through *Q*, as in the other pump. Then when the water follows *Q* as *Q* aſcends, upon it's deſcent it ſhuts the valve at *P* by preſſing upon it, and opens that at *R* and forces out the water.

142. In this pump, *Q* muſt, at it's higheſt point, be within 32 feet of the water in the reſervoir *ABCD*, becauſe in the rareſt ſtate of the atmoſphere, the preſſure of the air will not raiſe the water in a vacuum above that altitude. In the other pump, *P* muſt be within a little leſs than the ſame diſtance, in order that the water may always riſe above it.

Prop. LXXI.

To explain the principle of the motion of water through a Syphon.

143. If one end of a Syphon *mno* be put into a veſſel of water, and the other end without be lower



than the ſurface of the water; then if the air be drawn out, the water will begin and continue to run until the ſurface of the water in the veſſel is on a level with the end *o*.

144. For when the air is drawn out of the Syphon, the water will riſe in it to *n* by the preſſure of the air upon the ſurface of the water in the veſſel, and then it will deſcend to *o* by it's gravity. Now the preſſure of the air at *o* to force the water in the direction *onm*, is equal to the preſſure of the air on the ſurface of the fluid in the veſſel to force the water in the direction *mno*, at leaſt extremely nearly ſo, on account of the very ſmall difference of the altitudes of the air above *m* and *o*; but the former preſſure is oppoſed by the preſſure of the column *no*, and the latter preſſure is oppoſed by the preſſure of the column *mn*; the latter preſſure of the air therefore being leſs oppoſed than the former preſſure, the fluid muſt move in the direction of the latter preſſure, or in the direction *mno*; and the fluid will continue to run till the preſſures of *on*, *mn* become equal, or till *o* and *m* are in the ſame horizontal line, for then their perpendicular heights being equal, their preſſures will be equal by Art. 31.

◆

SECTION X.

ON THE THERMOMETER, HYGROMETER, AND PYROMETER.

145. A THERMOMETER is an inſtrument conſtructed to meaſure different degrees of heat; it is a glaſs tube with a bulb at the bottom, having the bulb and part of the tube filled with a fluid; the tube is hermetically ſealed at the top, and the part not occupied by the fluid is a vacuum. Againſt the tube there is a ſcale to meaſure the expanſion of the fluid under different temperatures.

PROP. LXXII.

To find what Fluids are proper for Thermometers

146. Fluids expand by being heated, and contract again as they grow cold. Thoſe fluids, therefore, which are not ſubject to be frozen, and whoſe

expanſion is ſenſible and in proportion to the heat applied, are proper for thermometers. Now the expanſion of mercury, linſeed oil, and ſpirits of wine, is, as to ſenſe, proportional to the heat applied. This BROOK TAYLOR found by the following experiment. Having conſtructed a Thermometer with linſeed oil, he put it into cold water, and then into water heated to any degree, and noticed the altitudes at which the fluid in the Thermometer ſtood in each caſe. He then put equal quantities of theſe waters together, which gave a mean heat; and by putting the Thermometer into this mixture, he found that it ſtood at a mean altitude between the two former altitudes. And this appeared to be true of whatever temperatures the two parts of water were. The mean temperature, therefore always agreeing with the mean altitude, the expanſion muſt be in proportion to the heat. The ſame is found true of mercury and of ſpirits of wine.

PROP. LXXIII.

To fill a Thermometer.

147. The bore of the tube is ſo ſmall that the fluid cannot be poured in; to get in the fluid, therefore, heat the bulb, by blowing the flame of a lamp againſt it with a blow-pipe, and you will expel the air from within; then dip the open end of the tube into the fluid, and it will riſe up into the tube and bulb, by the preſſure of the air upon the ſurface of the fluid into which you dip it, there being a vacuum, or nearly ſo, within the tube and bulb. If it do not fill the firſt time, repeat the operation till it does;

and if there be any air bubbles, tie a ſtring to the end of the tube and whirl it about till the bubbles eſcape. Having thus filled the tube, hold it over the lamp till it boils, and in that ſtate, let it be hermetically ſealed, and upon the deſcent of the fluid, when it grows cold, the ſpace above muſt be a vacuum.

PROP. LXXIV.

To graduate a Thermometer according to FAHRENHEIT's *ſcale.*

148. Having filled the tube of the Thermometer, and fixed it againſt a frame upon which the graduations are to be made, put it into water juſt freezing, and againſt the ſurface put 32; then put it into boiling water, and againſt the fluid put 212; divide this interval into 180 equal parts, and alſo continue the ſame diviſions down below 32 to the bulb. Then will 98 be blood heat, 76 ſummer heat, and 55 temperate. If the tube and ſcale be continued upwards to 600, it will give the heat of boiling mercury; and if it be continued downwards to 40 below 0, it will give the cold of freezing mercury. Or a thermometer may be graduated by comparing it with another, in this manner. Put them both into water, firſt of one temperature and then of another, and mark the ungraduated one in theſe two caſes, according to the graduated one; then this interval may be ſubdivided, and the graduation continued both ways.

149. Hence a thermometer may be graduated for any other ſcale. In SIR I. NEWTON's ſcale, freezing water is 0 and boiling water 34; and the other points

may be found by proportion from the other ſcale. For inſtance, to find blood heat on this ſcale, we may obſerve, that in Fahrenheit's thermometer, from freezing to boiling water is 180, and to blood heat 66; and in this ſcale, from freezing to boiling water is 34; hence 180 : 66 :: 34 : $12\frac{7}{15}$ the point of blood heat on this ſcale.

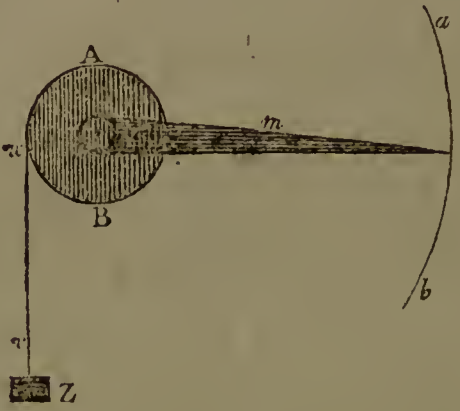
150. The preſſure of the atmoſphere againſt the outſide of the bulb, not being counteracted by any air within, affects it's magnitude, diminiſhing it as the preſſure is increaſed. The variation however which this cauſes on the ſcale is never above one tenth of a degree. Thermometers are generally made with ſpirits of wine or mercury, becauſe linſeed oil is found to adhere to the ſides of the tube, which prevents it from ſhowing ſuddenly any change of temperature. It is better to make the bulb flat than globular, becauſe all the fluid will then be ſooneſt affected by a variation of temperature.

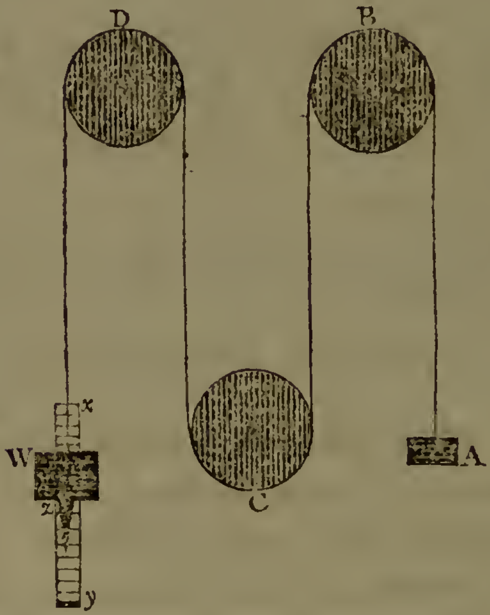
Prop. LXXV.

To conſtruct an Hygrometer.

151. An Hygrometer is an inſtrument to determine the degrees of moiſture and dryneſs of the air, and is formed by thoſe ſubſtances which will expand or contract upon any alteration of the moiſture. Wood expands by moiſture and contracts by dryneſs; on the contrary, chord, catgut, &c. contract by moiſture and expand by dryneſs. Various mechanical contrivances have been invented to render

ſenſible the ſmalleſt variations in the lengths of thoſe ſubſtances. We will deſcribe two of them.—Let *AB* be the ſection of a cylinder moveable about it's axis, which is parallel to the horizon; at the end of which there is an index moveable againſt a graduated arc *ab*; about this cylinder ſome catgut is wound, one end of which is fixed to the cylinder, and the other end





to ſomething immoveable at *Z*. As the moiſture of the air increaſes, the catgut contracts and turns the cylinder, and the motion of the index ſhows the increaſe of the moiſture; and as the air decreaſes in moiſture, the catgut will lengthen, and the weight of the index will carry the cylinder back, and the index will ſhow the correſponding decreaſe of moiſture. In the ſecond figure, the catgut is fixed at *A* and goes over the pullies *B*, *C*, *D*, and at the other end a weight *W* is fixed, having an index *z* which moves againſt a graduated ſcale *xy*, that ſhows the increaſe and decreaſe of the length of the ſtring, and conſequently the ſtate of the air in reſpect to it's moiſture. Various other contrivances, upon the ſame principle, have been invented, but it would be foreign to the plan of this work to enter into a particular deſcription of every inſtrument which has been conſtructed for this purpoſe.

152. Mr. De Luc has made a great many experiments, in order to find out ſuch ſubſtances as expand moſt nearly in proportion to the quantity of moiſture imbibed. The reſult was, that whalebone and box, cut acroſs the fibres, increaſe very nearly in proportion to the quantity of moiſture, and more nearly ſo than any other ſubſtances which he tried. This he found by taking a quantity of ſhavings of each ſubſtance, and weighing them at the time when he meaſured the increaſe of the length of a ſlip of each, cut as above deſcribed, the increaſe of weight being always in proportion to the increaſe of length. In his conſtruction of an Hygrometer he preferred the whalebone, firſt, on account of it's ſteadineſs, in always coming to the ſame point at extreme moiſture; ſecondly, on account of it's greater expanſion, it in-

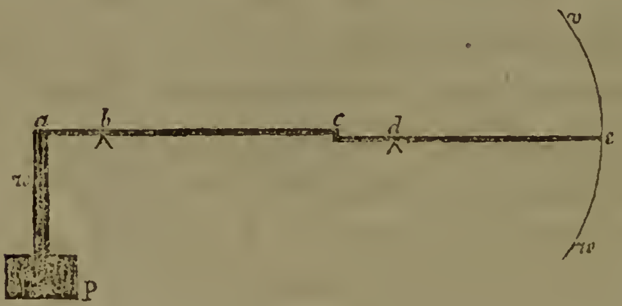
creaſing in length above one eighth of itſelf from extreme dryneſs to extreme moiſture; laſtly, it is more eaſily made thin and narrow.

It is a little extraordinary, that when he took threads of ſome ſubſtances in the direction of the fibres, they firſt increaſed as the quantity of moiſture increaſed, and afterwards upon a further increaſe of moiſture they decreaſed in length. See the *Phil. Tranſ.* for 1791.

PROP. LXXVI.

To conſtruct a Pyrometer.

153. A *Pyrometer* is an inſtrument invented to ſhow the expanſion and contraction of metals by heat and cold. Various machines have been conſtructed for this purpoſe; but as it would not be conſiſtent with the plan of this work to enter into a particular deſcription of each, we ſhall here only explain the general principle. Let *abc* be a lever whoſe fulcum is *b*, acting upon another lever *cde*, whoſe fulcum is *d*; and let *w* be a metalic rod, one end of which reſts againſt an immoveable obſtacle P, and the other end againſt the lever *abc* at *a*. If a lamp



lamp be put under this rod, the heat will increaſe it's length, and put the levers in motion; now

$$\begin{array}{rl} & \text{vel. of } a : \text{vel. of } c :: ab : bc \\ & \text{vel. of } c : \text{vel. of } e :: cd : de \\ \hline \therefore & \text{vel. of } a : \text{vel. of } e :: ab\times cd : bc\times de \end{array}$$

Hence if bc and de be very great in proportion to ab and cd, a ſmall increaſe in the length of w will produce a conſiderable motion in the point e, which may be meaſured upon the graduated arc vw.

For example, if $ab : bc :: 1 : 25$, and $cd : de :: 1 : 40$, then $ab\times cd : bc\times de :: 1\times 1 : 25\times 40 :: 1 : 1000$; hence whilſt the rod increaſes the 1000th part of an inch, the end e will deſcribe 1 inch. On this principle the leaſt increaſe of the length of the rod becomes viſible. Inſtead of putting the lamp immediately under the rod w, this rod is laid upon another piece of metal, called the heater, and when the lamp has given this it's greateſt degree of heat, the rod w is laid upon it.

154. In this manner Mr. MUSCHENBROEK made experiments to determine the proportion of the expanſions of different metals, by applying a different number of lamps, and found the reſult as follows:

Lamps.	Iron.	Steel.	Copper.	Braſs.	Tin.	Lead.
1	80	85	89	110	153	155
2	117	123	115	220	*	274
3	142	168	193	275	*	*
4	211	270	270	361	*	*
5	230	310	310	377	*	*

Tin melted with two lamps and lead with three. With this kind of pyrometer Mr. FERGUSON found the expanſion of metals to be in the following proportion; iron and ſteel 3, copper 4½, braſs 5, tin 6, lead 7. An iron rod 3 feet long is about one 70th of an inch longer in ſummer than in winter.

155. If a metal be put into water and the water be heated, the metal expands as the heat of the water increaſes. By this method Mr. SMEATON determined the expanſion of different metals; for by means of a mercurial thermometer immerſed in the water he could always aſcertain the degree of heat. He found that in equal intervals of time the expanſions were in geometric progreſſion. By this he was enabled to get the meaſure of the bar before it was applied to the inſtrument. This will be beſt underſtood by explaining an experiment. The time elapſed between applying the bar to the inſtrument and taking the firſt meaſure, was ½ a minute; therefore the intervals between taking the ſucceeding meaſures were ½ a minute alſo. The firſt meaſure was 208; the ſecond 214,5; the third 216,5; the fourth 217,5. The differences of theſe are 6,5; 2; 1. Now theſe three numbers are nearly equal to 6, 3; 2, 25; 0, 8, which form a geometrical progreſſion whoſe common ratio is 2, 8. As therefore we may ſuppoſe the expanſion from the inſtant the bar was applied to the time of taking the firſt meaſure followed the ſame law, we can find the expanſion in the firſt ½ minute (at the end of which the firſt meaſure was taken) by continuing back the progreſſion, or multiplying 6,3 by 2,8, which gives 17,7 for the expanſion the firſt ½ minute; hence 208 − 17,7 = 190,3 for the meaſure

before the bar was applied. The following expanſions are ſelected from Mr. Smeaton's table, ſhowing how much a foot, in length, of each increaſes in decimals of an inch, by an increaſe of heat correſponding to 180 degrees of Fahrenheit's thermometer from freezing to boiling water. See Mr. Smeaton's account in the *Phil. Tranſ.* 1754.

White glaſs barometer tube	,01
Hard ſteel	,0147
Iron	,0151
Copper hammered	,0204
Caſt braſs	,0225
Grain tin	,0298
Lead	,0344
Zinc	,0353

156. Metals being thus ſubject to expanſion by heat, a pendulum made with a ſingle rod of metal will continually be ſubject to a variation in it's length from the variation of the temperature of the air. To correct this, Mr. Harrison invented a pendulum, called a gridiron pendulum, compoſed of rods of ſteel and rods of braſs, ſo connected together, that the braſs expands upwards when the ſteel expands downwards; and by a proper adjuſtment of theſe rods, the diſtance from the point of ſuſpenſion to the center of oſcillation, may be rendered ſubject but to a very ſmall variation. Mr. Graham invented another method of preſerving the length of the pendulum the ſame in different temperatures. He took a glaſs, or metallic tube, and put in ſome mercury; and the

heat, which expands the glaſs or metal downwards, expands the mercury upwards; by the adjuſtment therefore of a proper quantity of mercury, he could make theſe effects in altering the length of the pendulum nearly deſtroy each other. He found the errors of a clock of this ſort to be only about $\frac{1}{8}$ of the errors of the beſt clock of the common ſort.

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SECTION XI.

ON WINDS, SOUND, VAPOURS, AND THE FORMATION OF SPRINGS.

PROP. LXXVII.

To explain the Cauſes of the various Winds.

157. WIND is a current of air, and it's direction is denominated from that point of the compaſs *from* which it blows. The principal, if not the only cauſe of wind is a partial rarefaction of the air by heat. When the air is heated, it becomes rarer, and therefore aſcends; and the ſurrounding cold air ruſhing in to ſupply it's place, forms a current in ſome one direction. Winds may be divided into *conſtant*, or thoſe which always blow in the ſame direction; *periodical*, or thoſe which blow half a year in one direction, and half a year in the contrary direction; theſe are called *monſoons*; and *variable*, which are ſubject to no rules. The two former are alſo called *Trade* winds. We ſhall here give the principal phœnomena of winds, from

Dr. HALLEY's Hiſtory thereof, in the *Philoſophical Tranſactions.*

1ſt. In the *Atlantic* and *Pacific Ocean*, under the Equator there is a conſtant Eaſt wind.

2ly. To about 28° on each ſide of the Equator, the wind on the *north* ſide declines towards the north eaſt; and the more ſo, the further you recede from the Equator; and on the *ſouth* ſide, it declines in like manner towards the ſouth eaſt. The limits of theſe winds are greater in the Pacific Ocean, on the American, then on the African ſide, extending in the former caſe to about 32°, and in the latter to about 28°. And this is true likewiſe to the ſouthward of the Equinoctial, for near the *Cape of Good Hope*, the limits of the trade winds are 3° or 4° nearer the line, than on the coaſt of *Braſil.*

3ly. Towards the *Caribbee* Iſlands, the aforeſaid north-eaſt wind becomes more and more eaſterly, ſo as ſometimes to be eaſt, and ſometimes eaſt by ſouth, but moſtly northwards of the eaſt, a point or two.

4ly. On the coaſt of Africa, from the *Canaries* to about 10°. N. latitude, the wind ſets in towards the north weſt; then it becomes ſouth weſt, approaching more to the ſouth as you approach the *Cape*. But away from the coaſts, the winds are perpetually between the ſouth and the eaſt; on the African ſide they are more ſoutherly; on the Braſilian, more eaſterly, ſo as to become almoſt due eaſt. Upon the coaſt of Guinea, they are ſubject to frequent calms, and violent ſudden guſts, called *Tornado's*, from all points of the compaſs.

5ly. In the *Indian* ocean, the winds are partly *conſtant*, and partly *periodical.* Between *Madagaſcar* and *New Holland*, from 10° to 30° latitude, the wind blows ſouth-eaſt by eaſt. During the months of *May*, *June*, *July*, *Auguſt*, *September*, *October*, the aforeſaid ſouth-eaſt winds extend to within 2° of the Equator; then for the other ſix months, the contrary winds ſet in, and blow from 3° to 10° S. latitude. From 3° S. latitude over the *Arabian* and *Indian* ſeas and *Bay of Bengal*, from Sumatra to the coaſt of Africa, there is another monſoon, blowing from *October* to *April* on the north-eaſt point, and in the other half year from the oppoſite direction. Between *Madagaſcar* and *Africa*, a ſouth-ſouth-weſt wind blows from *April* to *October*, which, as you go more northerly, becomes more weſterly, till it falls in with the weſt-ſouth-weſt winds; but the Dr. could not obtain a ſatisfactory account, how the winds are in the other half year. To the eaſtward of *Sumatra* and *Malacca*, on the north ſide of the Equator along the coaſt of *Cambodia* and *China*, the monſoons blow and change at the ſame time as before-mentioned; but their directions are more northerly and ſoutherly. Theſe winds reach to the *Philippine Iſlands* and to *Japan.* Between the ſame Meridians, on the ſouth ſide of the Equator, from *Sumatra* to *New Guinea*, the ſame monſoons are obſerved. The ſhifting of theſe winds is attended with great hurricanes.

158. The eaſt wind about the Equator is thus explained. The ſun moving from eaſt to weſt, the point of greateſt rarefaction of the air, by the heat of the ſun, muſt move in the ſame direction; and

the point of greateſt rarefaction following the ſun, the air muſt continually ruſh in from the caſt and make a conſtant eaſt wind.

159. The conſtant north-eaſt wind on the north ſide of the Equator, and ſouth-eaſt wind on the ſouth ſide, may be thus accounted for. The air towards the poles being denſer than that at the Equator, will continually ruſh towards the Equator; but as the velocity of the different parts of the earth's ſurface, from it's rotation, increaſes as you approach the Equator, the air which is ruſhing from the north towards the Equator will not continue upon the ſame meridian, but it will be left behind; that is, in reſpect to the earth's ſurface, it will have a motion from the eaſt, and theſe two motions combined produce a north-eaſt wind on the north ſide of the Equator. And in like manner, there muſt be a ſouth-eaſt wind on the ſouth ſide. The air which is thus continually moving from the Poles to the Equator, being rarefied when it comes there, aſcends to the top of the atmoſphere, and then returns back to the poles.

160. The cauſe of the *periodical* winds is ſuppoſed to be owing to the courſe of the ſun northward and ſouthward of the Equator. Dr. HALLEY explains them thus, "Seeing that ſo great Continents do interpoſe and break the continuity of the Ocean, regard muſt be had to the nature of the ſoil and the poſition of the high mountains, which I ſuppoſe the two principal cauſes of the ſeveral variations of the winds, from the former general rule: for if a country lying near the ſun prove to be flat, ſandy,

low land, ſuch as the *Deſarts of Libya* are uſually reported to be, the heat occaſioned by the reflection of the ſun's beams, and the retention thereof in the ſand, is incredible to thoſe that have not felt it; whereby the air being exceedingly rarefied, it is neceſſary that the cooler and more denſe air ſhould run thitherward to reſtore the equilibrium. This I take to be the cauſe, why near the coaſt of *Guinea* the wind always ſets in upon the land, blowing weſterly inſtead of eaſterly, there being ſufficient reaſon to believe, that the inland parts of *Africa* are prodigiouſly hot, ſince the northern borders thereof were ſo intemperate, as to give the antients cauſe to conclude, that all beyond the *Tropic* was made uninhabitable by exceſs of heat. From the ſame cauſe it happens, that there are ſo conſtant calms in that part of the ocean, called the *rains*. For this tract being placed in the middle, between the weſterly winds blowing on the coaſt of *Guinea*, and the eaſterly trade winds blowing to the weſtwards thereof, the tendency of the air here is indifferent to either, and ſo ſtands in equilibrio between both; and the weight of the incumbent atmoſphere being diminiſhed by the continual contrary winds blowing from hence, is the reaſon that the air here holds not the copious vapour it receives, but lets it fall into ſo frequent rains.

As the cool and denſe air, by reaſon of its greater gravity, preſſes upon the hot and rarefied, 'tis demonſtrative that this latter muſt aſcend in a continual ſtream as faſt as it is rarefied, and that being aſcended it muſt diſperſe itſelf to preſerve the æquilibrium,

librium, that is, by a contrary current, the upper air muſt move from thoſe parts where the greateſt heat is: So by a kind of circulation, the N. E. trade wind below, will be attended with a S. W. above, and the S. E. with a N. W. wind above. And that this is more than a bare conjecture, the almoſt inſtantaneous change of the wind to the oppoſite point, which is frequently found in paſſing the limits of the trade winds, ſeems to aſſure us; but that which above all confirms this hypotheſis is the phænomenon of the monſoons, by this means moſt eaſily ſolved, and without it hardly explicable. Suppoſing therefore ſuch a circulation as above, 'tis to be conſidered that to the northward of the *Indian Ocean* there is every where land within the uſual limits of the latitude of 30°, viz. *Arabia*, *Perſia*, *India*, &c. which for the ſame reaſon as the mediterranean parts of *Africa* are ſubject to unſufferable heats when the ſun is to the north, paſſing nearly vertical, but yet are temperate enough when the ſun is removed towards the other tropic; becauſe of a ridge of mountains at ſome diſtance within the land, ſaid to be frequently in winter covered with ſnow, over which the air, as it paſſes, muſt needs be much chilled. Hence it comes to paſs, that the air coming, according to the general rule, out of the N. E. in the *Indian* ſeas, is ſometimes hotter, ſometimes colder than that which by this circulation is returned out of the S. W. and by conſequence, ſometimes the under current or wind is from N. E. ſometimes from the S. W. as is clear from the times wherein theſe winds ſet in, viz. in *April*, when the ſun begins to warm thoſe countries to the north, the S. W. monſoon begins, and blows during the heats till *October*, when the ſun being retired, and all things growing cooler northward, and the heat increaſing to the ſouth, the N. E. winds enter and blow all the winter till *April* again.

And it is undoubtedly from the ſame principle that to the ſouthward of the Equator, in part of the *Indian Ocean*, the N. W. wind ſucceeds the S. E. when the ſun draws near the Tropric of *Capricorn*. But I muſt confeſs, that in this latter occurs a difficulty not well to be accounted for, which is, why this change of the monſoons ſhould be any more in this Ocean, than in the ſame latitudes in the Ethiopic, where there is nothing more certain than a S. E. wind all the year.

'Tis likewiſe very hard to conceive, why the limits of the trade-wind ſhould be fixt about the 30th deg. of latitude all round the globe; and that they ſhould ſo ſeldom tranſgreſs or fall ſhort of thoſe bounds; as alſo that in the *Indian* ſea, only the northern part ſhould be ſubject to the changeable monſoons, and in the ſouthern there be a conſtant S. E."

161. We may further add, that the cauſes mentioned in the laſt article, muſt here alſo operate. There may perhaps be ſome caſes of theſe periodical winds, which we cannot ſee altogether a correct ſolution of; but if all the circumſtances of ſituation, heat, cold, &c. were known, there is no reaſon to doubt but that they might be accounted for from the principles here delivered

162. We may further obſerve in reſpect to the direction in which winds blow, that if a current ſet off in any one direction, north-eaſt for inſtance, and move in a great circle, it will not continue to move on that point of the compaſs, becauſe a great circle will not meet all the meridians at the ſame angle, the meridians not being parallel. This circumſtance muſt therefore enter into our conſideration in eſtimating the direction of the wind. High mountains are alſo obſerved to turn the winds into a particular courſe. On the lake of *Geneva*, there are only two winds, that is, either up or down the valley. And the like is known to happen at other ſuch places.

163. The *conſtant* and *periodical* winds blow only at ſea; at land, the wind is always *variable*.

164. Beſides the winds already mentioned, there are others called *Land* and *Sea Breezes*. The air over the land being hotter during the day, than the air over the ſea, a current of air will ſet in from the ſea to the land by day; but the air over the land being colder than that over the ſea at night, the current at that time will be from the land to the ſea. This is very remarkable in Iſlands ſituated between the tropics.

165. Mr. CLARE exemplifies this by the following experiment. In the middle of a veſſel of water, place a water-plate of warm water, the water in the veſſel repreſenting the ocean, and the plate, the iſland rarefying the air over it. Then hold a lighted candle over the cold water, and blow it out, and the ſmoke will move towards the plate.

But if the plate be cold, and the ambient fluid be warm, the ſmoke will move in the contrary direction.

166. Dr. DERHAM, from repeated obſervations upon the motion of light, downy feathers, found that the greateſt velocity of wind was not above 60 miles in an hour. But Mr. BRICE juſtly obſerves, that ſuch experiments muſt be ſubject to great inaccuracy, as the feathers cannot proceed in a ſtraight line; he therefore eſtimates the velocity by means of the ſhadow of a cloud over the earth; by which he found, that in a great ſtorm the wind moved 63 miles in an hour; when it blows a freſh gale, at the rate of 21 miles an hour; and in a ſmall breeze, at the rate of about 10 miles in an hour. But this method takes for granted, that the clouds move as faſt as the wind. It is probable that the velocity is ſomething more than what is here ſtated.

PROP. LXXVIII.

To explain the Nature of Sound.

167. *Sound* is a ſenſation excited by the vibrations of the air upon the tympanum or drum of the ear. That the air is the inſtrument by which ſound is conveyed from the ſonorous body is manifeſt from hence, that no ſound can be produced if the body be in a vacuum, or if there be a vacuum between the body and the ear.

168. By percuſſion, the parts of a ſonorous body, as a bell, a muſical ſtring, &c. are put into a ſtate of

vibration, and as long as the vibrations are continued, correſponding vibrations are communicated to the air; and ſound is heard, as long as the vibrations are ſtrong enough to produce the ſenſation. All ſonorous bodies are therefore elaſtic. The manner in which the vibrations are excited in the air is ſo clearly deſcribed by Mr. Cotes, that I cannot do better than give the account in his own words. "The parts of the ſonorous body, being put into a tremulous and vibrating motion, are by turns moved forwards and backwards. Now as they go forwards they muſt of neceſſity preſs upon the parts of the air to which they are contiguous, and force them alſo to move forwards in the ſame direction with themſelves; and conſequently thoſe contiguous parts will at that time be condenſed; then as the parts of the ſonorous body return back again, the parts of the air which were juſt before condenſed, will be permitted to return with them, and by returning they will again expand themſelves. It is manifeſt therefore, that the contiguous parts of the air will go forwards and backwards by turns, and be ſubject to the like vibrating motion with the part of the ſonorous body.

"And as the ſonorous body produces a vibrating motion in the contiguous parts of the air, ſo will theſe parts thus agitated, in like manner produce a vibrating motion in the next parts, and thoſe in the next, and ſo on continually. And as the firſt parts were condenſed in their progreſs, and relaxed in their regreſs, ſo will the other parts, as often as they go forwards, be condenſed, and as often as they go backwards, be relaxed. And therefore they will not all go forwards together, and all go backwards together; for then their reſpective diſtances would always be the ſame, and conſequently they could not be rarefied and condenſed by turns; but meeting each other when they are condenſed, and going from each other when they are rarefied, they muſt neceſſarily one part of them go forwards whilſt the other goes backwards, by alternate changes from the firſt to the laſt.

"Now the parts which go forwards, and by going forwards are condenſed, conſtitute thoſe pulſes which ſtrike upon our organs of hearing, and other obſtacles they meet with; and therefore a ſucceſſion of pulſes will be propagated from the ſonorous body. And becauſe the vibrations of the ſonorous body follow each other at equal intervals of time, the pulſes which are excited by thoſe ſeveral vibrations, will alſo ſucceed each other at the ſame equal intervals."

169. As, when a fluid is put in motion, that motion is communicated in all directions, Sound muſt be propagated in all directions from a ſonorous body as a center, in concentric ſuperficies, or ſhells of air, called *Aeriel Pulſes*, or *Waves of Air*, analogous, as ſuppoſed by ſome, to the circular waves produced on the ſurface of water when a ſtone is thrown in. If the ſound be impeded by a body which has a hole, the waves paſs through, and diverge from it as a new center, and the ſound is heard on all parts on the other ſide of the body.

170. The law by which the force of ſound decreaſes as you recede from the ſonorous body, is not eaſy to be determined by theory. It has been uſually

eſtimated, by dividing the ſurrounding air into ſhells of an equal thickneſs, and ſuppoſing theſe ſhells to act upon each other, as ſo many elaſtic bodies would; but it is probable that this is a ſuppoſition very far diſtant from the truth. The utmoſt diſtance at which a ſound has been heard, is about 200 miles. This was obſerved in the war between England and Holland, in the year 1672. The unaſſiſted human voice has been heard from Old to New Gibraltar, a diſtance of 10 or 12 miles; the watch-word, *All's well*, given at the latter, in a ſtill night, having been heard at the former. In both theſe caſes, the ſound paſſed over the water; and it is found, that ſound will always be conveyed much further along a ſmooth, than a rough ſurface.

171. The velocity of ſound, produced by all bodies, is found by experiment to be 1142 feet in a ſecond, ſubject to a ſmall variation from the courſe of the wind. Dr. DERHAM determined this very accurately, by placing cannon at different diſtances, and firing them, and obſerving the interval between the flaſh and the report. And thus he alſo found that ſound (or rather the pulſes of air which excite it) moves uniformly; it being always found, that the interval was in proportion to the diſtance. Sir I. NEWTON determined the velocity of ſound by Theory; with which if the reader wiſh to be acquainted, he may conſult the PRINCIPIA, Lib. 2. Prop. 47. A very ſtrong wind is found to alter the velocity of ſound by about $\frac{1}{20}$ of the whole; to be added when the direction of the wind and ſound coincide, and ſubtracted, when they oppoſe each other.

172. Sound is conveyed to the greateſt diſtance by a trumpet, called a *ſpeaking* or *ſtentorophonic* trumpet, the form of which is like that figure which would be generated by any part of the logarithmic curve revolving about it's axis, the mouth being applied to the ſmaller end. The theory by which this is attempted to be proved, is ſubject to the objection mentioned in Art. 170.

173. The *ſame* ſound is always excited, when the air is put into the *ſame* ſtate of vibration; that is, if a bell and muſical ſtring make the air vibrate the *ſame number* of times in a ſecond, they excite the *ſame* tone. And as the ſame ſonorous body performs all it's vibrations in the ſame time, whether greater or leſs, the ſame body will always give the ſame tone, whether the percuſſive ſtroke be greater or leſs. The ſlower the vibration, the deeper or graver is the tone. But we mean not here to enter into the inveſtigation of the times of vibration of muſical ſtrings; a ſubject of conſiderable difficulty, and therefore not proper for an elementary treatiſe. If the reader wiſh for any information upon the ſubject, I refer him to Mr. PARKINSON's *Hydroſtatics*; or Dr. SMITH's *Harmonics*.

174. The reflection of the vibrations of the air from any fixed object to the ear, will cauſe a ſound diſtinct from that which is cauſed by the vibrations coming directly to the ear; and this is called an *Echo*. If the diſtance of the object which returns the echo be great, it will return ſeveral ſyllables. A ſingle ſyllable will not be clearly returned unleſs the diſtance of the object be at leaſt 120 feet; and ſo in proportion. Hence an echo returning ten ſyllables muſt

come from an object 1200 feet diſtant. More ſyllables however will be returned by night than by day, becauſe the air being then colder, is denſer, in which caſe, the return of the vibrations become ſlower, and conſequently more ſyllables may be heard.

PROP. LXXIX.

To explain the aſcent of Vapours, and the origin of Springs

175. *Vapours* are raiſed from the ſurface of the water; the principal cauſe of which is, probably, the heat of the ſun, the evaporation being always greateſt when the heat is the greateſt. The difficulty of ſolving the phænomenon ariſes from hence, that we find a heavier fluid (water) ſuſpended in a lighter fluid (air), contrary to our foregoing principles.

176. Dr. HALLEY ſuppoſed, that by the action of the ſun upon the ſurface of the water, the aqueous particles become formed into hollow bubbles filled with warm, and rarefied air, ſo as to make the whole bulk ſpecifically lighter than the air, in which caſe the particles will (Art. 45.) aſcend. But there is a great difficulty is conceiving how this can be effected. And if bubbles could be *at firſt* thus formed, when they aſcend, the air within would very ſoon be reduced to the ſame temperature of the air without, and they would immediately deſcend upon that effect taking place. Another opinion is, that the particles of water are ſeparated by a repulſive force, which is increaſed in proportion as the heat is increaſed, and thus they are diſperſed through the air; but the ſame

argument may be uſed againſt this hypotheſis, as againſt the laſt, that is, that this effect could not continue in the cold part of the atmoſphere where the clouds are ſuſpended. The moſt probable ſuppoſition is, that evaporation is a chemical ſolution of air in water. We know that metals are diſſolved in menſtruums, and their particles diffuſed and ſuſpended in the fluid, although their ſpecific gravity be greater than that of the fluid. Heat promotes this ſolution; in the day time therefore the heat cauſes a more perfect ſolution than what can, *cæteris paribus*, take place in the night when the air is colder, when the heat is frequently not ſufficient to keep the water in a ſtate of ſolution, and it falls in fogs and dews. The vapours, thus raiſed by heat, aſcend into the cold regions of the atmoſphere, and not being there kept in a ſtate of ſolution, they appear in the form of clouds; and when driven together by the agitation of the air, the particles run together into drops and fall down in rain. If they be frozen before they form themſelves into drops, they deſcend in ſnow; but if the drops of rain themſelves be frozen, they deſcend in hail. See HAMILTON on the *Aſcent of Vapours*.

177. MARRIOTTE ſuppoſed *Springs* to be owing to rain water and melted ſnow, which penetrating the ſurfaces of hills, and running by the ſide of clay or rocks which it cannot penetrate, at laſt comes to ſome place where it breaks out. This would account for the phænomenon, provided the ſupply from theſe cauſes was ſufficient; but D. SIDELEAU, and others, making an eſtimate of the quantity of rain and ſnow which falls in the ſpace of a year, to ſee whether it

would afford a quantity of water, equal to that which is annually diſcharged into the ſea by the rivers (which are ſupplied by ſprings), found that it would *not.* But Dr. Halley diſcovered the cauſe of a ſufficient ſupply; for he has proved by experiment, that the vapours which are raiſed, afford a much greater ſupply than is neceſſary; we will give the account in his own words.

178. "We took a pan of water (ſalted to the degree as is common ſea-water, by the ſolution of about a fortieth part of ſalt) about four inches deep, and 7 inches $\frac{9}{10}$ diameter, in which we placed a thermometer, and by means of a pan of coals, we brought the water to the ſame degree of heat which is obſerved to be that of the air in our hotteſt ſummers; the thermometer nicely ſhowing it. This done, we affixed the pan of water, with the thermometer in it, to one end of the beam of the ſcales, and exactly counterpoiſed it with weights in the other ſcale; and by the application or removal of the pan of coals, we found it very eaſy to maintain the water in the ſame degree of heat preciſely. Doing thus, we found the weight of the water ſenſibly to decreaſe; and at the end of two hours we obſerved, that there wanted half an ounce troy, all but 7 grains, or 233 grains of water, which in that time had gone off in vapour; tho' one could hardly perceive it ſmoak, and the water was not ſenſibly warm. This quantity in ſo ſhort a time ſeemed very conſiderable, being little leſs than 6 ounces in 24 hours, from ſo ſmall a ſurface as a circle of 8 inches diameter. To reduce this experiment to an exact *calculus*, and determine the thickneſs of the ſkin of water that had ſo evaporated, I aſſume

the experiment alledged by Dr. Edw. Bernard to have been made in the *Oxford Society*, viz. that the cube foot *Engliſh* of water weighs exactly 76 pounds troy; this divided by 1728, the number of inches in a foot, will give $253\frac{1}{3}$ grains, or half ounce $13\frac{1}{3}$ grains for the weight of a cube inch of water; wherefore the weight of 233 grains is $\frac{233}{253}$, or 35 parts of 38 of a cube inch of water. Now the area of the circle, whoſe diameter is $7\frac{9}{10}$ inches, is 49 ſquare inches; by which dividing the quantity of water evaporated, viz. $\frac{35}{38}$ of an inch, the quote $\frac{35}{1862}$ or $\frac{1}{53}$, ſhews that the thickneſs of the water evaporated was the 53d part of an inch: But we will ſuppoſe it only the 60th part, for the facility of calculation. If therefore water as warm as the air in ſummer, exhales the thickneſs of a 60th part of an inch in two hours from it's whole ſurface, in 12 hours it will exhale $\frac{1}{10}$ of an inch; which quantity will be found abundantly ſufficient to ſerve for all the rains, ſprings, and dews, and account for the *Caſpian Sea's* being always at a ſtand, neither waſting nor overflowing; as likewiſe for the current ſaid to ſet always in, at the *Straights* of *Gibraltar*, tho' thoſe *Mediterranean Seas* receive ſo many, and ſo conſiderable rivers.

179. "To eſtimate the quantity of water ariſing in vapours out of the ſea, I think I ought to conſider it only for the time the ſun is up, for that the dews return in the night as much, if not more vapours than are then emitted; and in ſummer the days being longer than

twelve hours, this exceſs is balanced by the weaker action of the ſun, eſpecially when riſing before the water is warmed: So that if I allow $\frac{1}{10}$ of an inch of the ſurface of the ſea to be raiſed *per diem* in vapours, it may not be an improbable conjecture.

" Upon this ſuppoſition, every 10 ſquare inches of the ſurface of the water yields in vapour *per diem* a cube inch of water; and each ſquare foot, half a wine pint; every ſpace of 4 feet ſquare, a gallon; a mile ſquare, 6914 tons; a ſquare degree, ſuppoſe of 69 *Engliſh* miles, will evaporate 33 millions of tuns: And if the *Mediterranean* be eſtimated at 40 degrees long and 4 broad, allowances being made for the places where it is broader by thoſe where it is narrower, (and I am ſure I gueſs at the leaſt) there will be 160 ſquare degrees of ſea; and conſequently the whole *Mediterranean* muſt loſe in vapour, in a ſummer's day, at leaſt 5280 millions of tuns. And this quantity of vapour, though very great, is as little as can be concluded from the experiment produced: And yet there remains another cauſe, which cannot be reduced to the rule, I mean the winds, whereby the ſurface of the water is licked up, ſomewhat faſter than it exhales by the heat of the ſun, as it is well known to thoſe that have conſidered thoſe drying winds which blow ſometimes.

" The *Mediterranean* receives theſe conſiderable rivers; the *Iberus*, the *Rhone*, the *Tiber*, the *Po*, the *Danube*, the *Nieſter*, the *Boryſthenes*, the *Tanais*, and the *Nile*, all the reſt being of no great note, and their quantity of water inconſiderable. We will ſuppoſe each of theſe nine rivers to bring down ten times as much water as the river *Thames*, not that any of them is ſo great in reality, but to comprehend with them all the ſmall rivulets that fall into the ſea, which otherwiſe I know not how to allow for.

"To caculate the water of the *Thames*, I aſſume that at *Kingſton Bridge*, where the flood never reaches, and the water always runs down, the breadth of the channel is 100 yards, and it's depth 3, it being reduced to an equality, (in both which ſuppoſitions I am ſure I take with the moſt.) Hence the profile of the water in this place is 300 ſquare yards: This multiplied by 48 miles, (which I allow the water to run in 24 hours, at 2 miles in an hour) or 84480 yards, gives 25344000 cubic yards of water to be evacuated every day, that is 20300000 tons *per diem*; and I doubt not but in the exceſs of my meaſure of the channel of the river, I have made more than ſufficient allowance for the waters of the *Brent*, the *Wandel*, the *Lea*, and *Darwent*, which are all worth notice, that fall into the *Thames* below *Kingſton*.

"Now if each of the aforeſaid nine rivers yield ten times as much water as the *Thames* doth, 'twill follow that each of them yields but 203 millions of tons *per diem*, and the whole nine but 1827 millions of tons in a day; which is but little more than $\frac{1}{3}$ of what is proved to be raiſed in vapours out of the *Mediterranean* in twelve hours time."

180. Beſides the *Conſtant* Springs, there are others which *ebb* and *flow* alternately, which may thus be accounted for. The water, before it breaks out, may meet with a large cavity on the ſide of the hill, and

the water, upon the overflowing of this reſervoir, may find an aperture, and make it's eſcape; in caſe of dry weather, therefore, the ſupply of water may not be ſufficient to keep it full, in which caſe, the ſpring will ceaſe to flow, and continue dry, till a ſupply cauſes it overflow, and produce again the ſpring.

THE END OF VOL III.

◆

ERRATA.

Page 3, line 22, for *eaſiy*, read *eaſy*.
12, line 9, for AB, read AD.
15, line 18, for $o \times z$, read $o \times oz$.
19, line 11, for (*Art.* 17), read (*Art.* 18).
26, line 14, for $W = 5787$, read $W = ,5787$.
28, line 16, for (*Art.* 39), read (*Art.* 38).
33, line 5, for (*Art.* 43), read (*Art.* 42).
35, line 1, for H, read P.
45, line 4, for BCD, read *ſin.* BCD.
48, line 6, for Cm^2, read Pm^2.
75, line 5, for a, read c.
76, laſt line, for $a : b :: b :: c : c :: d : d ::$ &c.
read $a : b :: b : c :: c : d :: d :$ &c.
77, line 19, for (Prop. 57), read (Prop. 58).
128, line 10, for *Pacific*, read *Atlantic*.